

## Gluon tree amplitudes in open twistor string theory

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

JHEP12(2009)032

(<http://iopscience.iop.org/1126-6708/2009/12/032>)

[The Table of Contents](#) and [more related content](#) is available

Download details:

IP Address: 80.92.225.132

The article was downloaded on 01/04/2010 at 13:20

Please note that [terms and conditions apply](#).

## Gluon tree amplitudes in open twistor string theory

---

**Louise Dolan<sup>a</sup> and Peter Goddard<sup>b</sup>**

<sup>a</sup>*Department of Physics, University of North Carolina,  
Chapel Hill, NC 27599, U.S.A.*

<sup>b</sup>*Institute for Advanced Study,  
Princeton, NJ 08540, U.S.A.*

*E-mail:* [ldolan@physics.unc.edu](mailto:ldolan@physics.unc.edu), [pgoddard@ias.edu](mailto:pgoddard@ias.edu)

ABSTRACT: We show how the link variables of Arkani-Hamed, Cachazo, Cheung and Kaplan (ACCK), can be used to compute general gluon tree amplitudes in the twistor string. They arise from instanton sectors labelled by  $d$ , with  $d = n - 1$ , where  $n$  is the number of negative helicities. Read backwards, this shows how the various forms for the tree amplitudes studied by ACCK can be grouped into contour integrals whose structure implies the existence of an underlying string theory.

KEYWORDS: Duality in Gauge Field Theories, Conformal Field Models in String Theory, Extended Supersymmetry, String Duality

ARXIV EPRINT: [0909.0499](https://arxiv.org/abs/0909.0499)

---

**Contents**

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>The <math>N</math>-point amplitude</b>	<b>3</b>
<b>3</b>	<b>The amplitude as an integral over constraints</b>	<b>6</b>
<b>4</b>	<b>The form of the integrand, <math>F(c)</math></b>	<b>8</b>
<b>5</b>	<b>Parameterization of the general solution of linear constraints on <math>c_{ir}</math></b>	<b>11</b>
<b>6</b>	<b>NMHV 6-point functions</b>	<b>15</b>
<b>7</b>	<b>Multiple constraints and an NMHV 7-point function</b>	<b>19</b>
<b>A</b>	<b>Interchange of particles between <math>\mathcal{P}</math> and <math>\mathcal{N}</math></b>	<b>25</b>
<b>B</b>	<b>Relations for the 6-point function</b>	<b>26</b>
<b>C</b>	<b>Relations for the 7-point function</b>	<b>26</b>

---

**1 Introduction**

Inspired by work of Arkani-Hamed, Cachazo, Cheung and Kaplan [1, 2], we use link variables to obtain expressions for tree amplitudes in open twistor string theory [3]–[5]. This extends the derivation from the twistor string beyond maximally helicity violating (MHV) amplitudes and special cases of non-MHV trees [6]–[12], as well as providing a basis for the dual structure envisioned in ACCK. We use a canonical quantization [13, 14] PROVA of Berkovits’ version of twistor string theory, and compute the gluon trees.

The extensive literature for amplitudes in the spinor helicity basis, sampled by [15]–[19], has been used in developing recursion relations [20, 21]. These were motivated by a remarkable formulation of string theory on twistor space [5], which made contact with a twistor description for gauge theory [22, 23]. Additional dual forms for trees are found [24]–[26].

At loop level, the twistor string has been difficult to interpret as a dual for the gauge theory [4, 13]. In this paper, however, we show that the gauge theory based analysis of ACCK, which is phrased in terms of link variables, appears naturally to lead back to the twistor string at tree level. This may eventually enlighten our treatment of string loops, and the pursuit of the dual S-matrix.

Suppose we have  $N$  gluons, labeled  $\alpha = 1, \dots, N$ , with momenta  $p_{\alpha\dot{\alpha}}^a = \pi_{\alpha}^a \bar{\pi}_{\alpha\dot{\alpha}}$ , and helicities  $\epsilon_{\alpha}$ ,  $m$  of which are positive and  $n$  negative,  $m + n = N$ . Write  $\mathcal{P}$  for the set of positive helicity particles and  $\mathcal{N}$  for the set of negative helicity particles.

The link variables  $c_{ir}, i \in \mathcal{P}, r \in \mathcal{N}$  satisfy the  $2N$  linear equations

$$\pi_j = \sum_{r \in \mathcal{N}} c_{jr} \pi_r \tag{1.1}$$

$$\bar{\pi}_s = - \sum_{i \in \mathcal{P}} \bar{\pi}_i c_{is}. \tag{1.2}$$

where we have suppressed the spinor indices. See [13] for our conventions. These equations are not independent because, as noted by ACCK, they imply momentum conservation. As in [1] (eq.(37)), these linear conditions imply energy-momentum conservation:

$$\sum_{j \in \mathcal{P}} \bar{\pi}_{j\dot{a}} \pi_j^{\dot{a}} = \sum_{j \in \mathcal{P}} \sum_{r \in \mathcal{N}} \bar{\pi}_{j\dot{a}} c_{jr} \pi_r^{\dot{a}} = - \sum_{r \in \mathcal{N}} \bar{\pi}_{r\dot{a}} \pi_r^{\dot{a}}; \tag{1.3}$$

for momenta satisfying this consistency condition they provide  $N' = 2N - 4$  constraints on the  $mn$  variables  $c_{ir}$ , leaving

$$N_R = mn - N' = (m - 2)(n - 2) \tag{1.4}$$

degrees of freedom. The philosophy outlined by ACCK is to seek to write the tree and loop amplitudes of gauge theory as contour integrals over the remaining  $N_R$  degrees of freedom.

The main observation underlying the analysis of this paper is that, in open twistor string theory, the link variables should be of the form

$$c_{js} = \frac{k_j}{k_s(\rho_j - \rho_s)}, \tag{1.5}$$

for some suitable  $\rho_\alpha, k_\alpha$ , as we shall see at the beginning of section 3. The necessary and sufficient condition for the link variables to be of this form is that the matrix

$$\begin{pmatrix} (c_{i_1 r_1})^{-1} & (c_{i_1 r_2})^{-1} & \dots & (c_{i_1 r_n})^{-1} \\ (c_{i_2 r_1})^{-1} & (c_{i_2 r_2})^{-1} & \dots & (c_{i_2 r_n})^{-1} \\ \vdots & \vdots & & \vdots \\ (c_{i_m r_1})^{-1} & (c_{i_m r_2})^{-1} & \dots & (c_{i_m r_n})^{-1} \end{pmatrix} \tag{1.6}$$

should have rank two, as we discuss in more detail in section 5. This is equivalent to the vanishing of the determinant of each  $3 \times 3$  submatrix, that is of each determinant of the form

$$\begin{aligned} \mathcal{C}_{rst}^{ijk} &= \begin{vmatrix} c_{is}c_{it} & c_{it}c_{ir} & c_{ir}c_{is} \\ c_{js}c_{jt} & c_{jt}c_{jr} & c_{jr}c_{js} \\ c_{ks}c_{kt} & c_{kt}c_{kr} & c_{kr}c_{ks} \end{vmatrix} \\ &= c_{ir}c_{js}c_{kt}c_{jr}c_{ks}c_{it} + c_{jr}c_{ks}c_{it}c_{kr}c_{is}c_{jt} + c_{kr}c_{is}c_{jt}c_{ir}c_{js}c_{kt} \\ &\quad - c_{kr}c_{js}c_{it}c_{jr}c_{is}c_{kt} - c_{jr}c_{is}c_{kt}c_{ir}c_{ks}c_{jt} - c_{ir}c_{ks}c_{jt}c_{kr}c_{js}c_{it}. \end{aligned} \tag{1.7}$$

For this condition to be met, it is sufficient for a suitable subset comprising  $N_R$  of the  $\mathcal{C}_{rst}^{ijk}$  to vanish, e.g. if we fix  $I, J \in \mathcal{P}$  and  $R, S \in \mathcal{N}$ , it is sufficient to have the vanishing of the  $N_R$  quantities  $\mathcal{C}_{RS\dot{t}}^{IJK}$  where  $k$  ranges over  $\mathcal{P}'$ , the remaining  $m - 2$  elements of  $\mathcal{P}$ , and  $t$

ranges over  $\mathcal{N}'$ , the remaining  $n - 2$  elements of  $\mathcal{N}$ . Using the linear conditions (1.1), (1.2) to express the  $c_{ir}$  in terms of the  $c_{kt}, k \in \mathcal{P}', t \in \mathcal{N}'$ , we find in section 3 that the tree amplitude will have the form

$$\oint F(c) \prod_{\substack{k \in \mathcal{P}' \\ t \in \mathcal{N}'}} \frac{dc_{kt}}{C_{kt}}, \quad (1.8)$$

where  $F(c)$  is a simple rational function of the  $c_{ir}$ .

In section 2, we review the derivation of vertex operator expressions for the general  $N$ -point tree amplitudes in twistor string theory from vertex operators. In section 3, we analyze the amplitude as an integral over constraints. In section 4, we derive the integrand function  $F(c)$ , as a function of the link variables, from twistor string theory. In section 5, we discuss the parametrization of the linear constraints, and complete the description of the contour integral expression for the amplitudes. In section 6, we compute all 6-point functions, including alternative forms, by evaluating the contour integral as a sum over residues. In section 7, we use our general formulae to check the 7-pt tree with alternating helicities.

## 2 The $N$ -point amplitude

As in [3, 13], we consider conjugate twistor variables  $Z$  and  $W$

$$Z = \begin{pmatrix} \pi^a \\ \omega^{\dot{a}} \end{pmatrix}, \quad W = \begin{pmatrix} \bar{\omega}_a \\ \bar{\pi}_{\dot{a}} \end{pmatrix}, \quad (2.1)$$

$$W \cdot Z = \bar{\omega} \cdot \pi + \bar{\pi} \cdot \omega \equiv \bar{\omega}_a \pi^a + \bar{\pi}_{\dot{a}} \omega^{\dot{a}}, \quad (2.2)$$

and the field describing the twistor string,

$$Z(\rho) = \begin{pmatrix} \lambda^a(\rho) \\ \mu^{\dot{a}}(\rho) \end{pmatrix}.$$

We fourier transform the open string vertex operators for gluons according to their helicity [1], as

$$V_+^A(W, \rho) = \int d^2 \pi^a e^{i\bar{\omega}_a \pi^a} \int \frac{d\kappa}{\kappa} \delta^2(\kappa \lambda^a(\rho) - \pi^a) e^{i\kappa \bar{\pi}_{\dot{b}} \mu^{\dot{b}}(\rho)} J^A = \int \frac{d\kappa}{\kappa} e^{i\kappa W \cdot Z(\rho)} J^A, \quad (2.3)$$

$$\begin{aligned} V_-^A(Z, \rho) &= \int d^2 \bar{\pi}_{\dot{a}} e^{-i\omega^{\dot{a}} \bar{\pi}_{\dot{a}}} \int \kappa^3 d\kappa \delta^2(\kappa \lambda^a(\rho) - \pi^a) e^{i\kappa \bar{\pi}_{\dot{b}} \mu^{\dot{b}}(\rho)} J^A \psi^1 \dots \psi^4 \\ &= \int \kappa^3 d\kappa \delta^4(\kappa Z(\rho) - Z) J^A \psi^1 \dots \psi^4. \end{aligned} \quad (2.4)$$

Defining  $X_j = W_j, j \in \mathcal{P}; X_s = Z_s, s \in \mathcal{N}$ , we compute the tree amplitudes as a sum over instanton sectors. The only non-vanishing contribution to any tree with  $n$  negative

helicity states is from the sector with instanton number  $d$ , where  $d = n - 1$ , [13]

$$\begin{aligned}
 M^{\epsilon_1 \dots \epsilon_N} &= \int \langle 0 | V_{\epsilon_1}^{A_1}(X_1, \rho_1) \dots V_{\epsilon_N}^{A_N}(X_N, \rho_N) | 0 \rangle \prod_{\alpha=1}^N d\rho_\alpha / dg \\
 &= \int \prod_{\alpha=1}^N \frac{d\rho_\alpha d\kappa_\alpha}{\kappa_\alpha} \langle 0 | e^{(n-1)q_0} \prod_{s \in \mathcal{N}} \delta^4(\kappa_s Z(\rho_s) - Z_s) \exp \left\{ i \sum_{j \in \mathcal{P}} \kappa_j W_j \cdot Z(\rho_j) \right\} | 0 \rangle \\
 &\quad \times \prod_{s \in \mathcal{N}} \kappa_s^4 \prod_{r < s; r, s \in \mathcal{N}} (\rho_r - \rho_s)^4 \langle 0 | J^{A_1}(\rho_1) J^{A_2}(\rho_2) \dots J^{A_N}(\rho_N) | 0 \rangle / dg \quad (2.5)
 \end{aligned}$$

and  $dg$  is the invariant measure on the group  $GL(2, \mathbb{R})$  of Möbius and scale transformations. Because  $Z(\rho)$  is a polynomial of order  $n - 1$ ,

$$Z(\rho) = \sum_{s \in \mathcal{N}} \frac{1}{\kappa_s} Z_s \prod_{r \neq s; r \in \mathcal{N}} \frac{\rho - \rho_r}{\rho_s - \rho_r}, \quad (2.6)$$

so that  $\kappa_r Z(\rho_r) = Z_r$ ,  $r \in \mathcal{N}$ . If  $\xi_r = \kappa_r Z(\rho_r) - Z_r$ ,  $r \in \mathcal{N}$ , the Jacobian resulting from performing the integrations corresponding to the zero modes of  $Z(\rho)$  is

$$\frac{\partial(\xi_r : r \in \mathcal{N})}{\partial(Z_{-n+1}, \dots, Z_0)} = \prod_{s \in \mathcal{N}} \kappa_s \prod_{r < s; r, s \in \mathcal{N}} (\rho_r - \rho_s).$$

This factor will cancel the worldsheet fermion contribution included in (2.5), and the amplitude becomes

$$M^{\epsilon_1 \dots \epsilon_N} = \int \prod_{\alpha=1}^N \frac{d\kappa_\alpha}{\kappa_\alpha} \exp \left\{ i \sum_{r \in \mathcal{N}} \sum_{j \in \mathcal{P}} c_{jr} W_j \cdot Z_r \right\} f^{A_1 A_2 \dots A_N} \left[ \prod_{\alpha=1}^N \frac{d\rho_\alpha}{\rho_\alpha - \rho_{\alpha+1}} \right] / dg, \quad (2.7)$$

where

$$c_{js} = \frac{\kappa_j}{\kappa_s} \prod_{r \neq s; r \in \mathcal{N}} \frac{\rho_j - \rho_r}{\rho_s - \rho_r}, \quad (2.8)$$

which we shall relate to (1.5) in section 3.

The action of the group  $GL(2, \mathbb{R})$  is defined by

$$\rho_\alpha \mapsto \frac{a\rho_\alpha + b}{c\rho_\alpha + d}, \quad \kappa_\alpha \mapsto (c\rho_\alpha + d)^{n-1} \kappa_\alpha, \quad (2.9)$$

which leaves  $c_{js}$  invariant.

We transform to momentum space by applying

$$\int \exp \left\{ -i \sum_{j \in \mathcal{P}} \bar{\omega}_j \cdot \pi_j + i \sum_{r \in \mathcal{N}} \bar{\pi}_r \cdot \omega_r \right\} \prod_{j \in \mathcal{P}} d^2 \bar{\omega}_j \prod_{r \in \mathcal{N}} d^2 \omega_r,$$

giving

$$\begin{aligned}
 M^{\epsilon_1 \dots \epsilon_N} &= f^{A_1 A_2 \dots A_N} \int \prod_{j \in \mathcal{P}} \delta^2 \left( \pi_j - \sum_{r \in \mathcal{N}} c_{jr} \pi_r \right) \prod_{s \in \mathcal{N}} \delta^2 \left( \bar{\pi}_s + \sum_{i \in \mathcal{P}} \bar{\pi}_i c_{is} \right) \\
 &\quad \times \left[ \prod_{\alpha=1}^N \frac{d\kappa_\alpha d\rho_\alpha}{\kappa_\alpha (\rho_\alpha - \rho_{\alpha+1})} \right] / dg. \quad (2.10)
 \end{aligned}$$

As we saw in section 1, the delta function conditions imply energy-momentum conservation. Fixing a choice of  $R, S \in \mathcal{N}$ ,

$$\begin{aligned} & \prod_{j \in \mathcal{P}} \delta^2 \left( \pi_j - \sum_{r \in \mathcal{N}} c_{jr} \pi_r \right) \prod_{s \in \mathcal{N}} \delta^2 \left( \bar{\pi}_s + \sum_{i \in \mathcal{P}} \bar{\pi}_i c_{is} \right) \\ &= \langle R, S \rangle^2 \delta^4 \left( \sum_{\alpha=1}^N \bar{\pi}_\alpha \pi_\alpha \right) \prod_{j \in \mathcal{P}} \delta^2 \left( \pi_j - \sum_{r \in \mathcal{N}} c_{jr} \pi_r \right) \prod_{s \in \mathcal{N}'} \delta^2 \left( \bar{\pi}_s + \sum_{i \in \mathcal{P}} \bar{\pi}_i c_{is} \right), \end{aligned} \quad (2.11)$$

so that we are left with  $2N - 4$  relations from the remaining delta functions to determine the  $2N - 4$  effective variables amongst the  $\kappa_\alpha, \rho_\alpha$ , after allowing for Möbius and scale invariance. The dependence of the  $mn = n(N - n)$  variables  $c_{ir}$  on the  $2N - 4$  effective variables  $\kappa_\alpha, \rho_\alpha$  implies the existence of

$$N_R = mn - (2N - 4) = (m - 2)(n - 2)$$

(nonlinear) relations between the  $c_{ir}$ , say  $\mathcal{C}_K(c) = 0, 1 \leq K \leq N_R$ . e.g. for  $N = 6, m = n = 3, \mathcal{P} = \{i, j, k\}, \mathcal{N} = \{r, s, t\}$ , there is one relation from (1.7):

$$\begin{aligned} & c_{ir} c_{js} c_{kt} c_{jr} c_{ks} c_{it} + c_{jr} c_{ks} c_{it} c_{kr} c_{is} c_{jt} + c_{kr} c_{is} c_{jt} c_{ir} c_{js} c_{kt} \\ &= c_{kr} c_{js} c_{it} c_{jr} c_{is} c_{kt} + c_{jr} c_{is} c_{kt} c_{ir} c_{ks} c_{jt} + c_{ir} c_{ks} c_{jt} c_{kr} c_{js} c_{it}. \end{aligned} \quad (2.12)$$

Now,

$$dg = \frac{d\rho_R d\rho_S}{(\rho_R - \rho_S)^2} \frac{d\kappa_R d\kappa_S}{\kappa_R \kappa_S},$$

so that

$$M^{\epsilon_1 \dots \epsilon_N} = f^{A_1 A_2 \dots A_N} \delta^4 \left( \sum_{\alpha=1}^N \bar{\pi}_\alpha \pi_\alpha \right) \mathcal{M}^{\epsilon_1 \dots \epsilon_N}, \quad (2.13)$$

with

$$\begin{aligned} \mathcal{M}^{\epsilon_1 \dots \epsilon_N} &= \langle R, S \rangle^2 (\rho_R - \rho_S)^2 \int \prod_{j \in \mathcal{P}} \delta^2 \left( \pi_j - \sum_{r \in \mathcal{N}} c_{jr} \pi_r \right) \prod_{s \in \mathcal{N}'} \delta^2 \left( \bar{\pi}_s + \sum_{i \in \mathcal{P}} \bar{\pi}_i c_{is} \right) \\ &\times \prod_{\alpha=1}^N \frac{1}{(\rho_\alpha - \rho_{\alpha+1})} \prod_{\substack{\alpha=1 \\ \alpha \neq R, S}}^N \frac{d\kappa_\alpha d\rho_\alpha}{\kappa_\alpha}. \end{aligned} \quad (2.14)$$

Note that if we chose  $I, J \in \mathcal{P}$  rather than  $R, S \in \mathcal{N}$ ,

$$\begin{aligned} & \prod_{j \in \mathcal{P}} \delta^2 \left( \pi_j - \sum_{r \in \mathcal{N}} c_{jr} \pi_r \right) \prod_{s \in \mathcal{N}} \delta^2 \left( \bar{\pi}_s + \sum_{i \in \mathcal{P}} \bar{\pi}_i c_{is} \right) \\ &= [I, J]^2 \delta^4 \left( \sum_{\alpha=1}^N \bar{\pi}_\alpha \pi_\alpha \right) \prod_{j \in \mathcal{P}'} \delta^2 \left( \pi_j - \sum_{r \in \mathcal{N}} c_{jr} \pi_r \right) \prod_{s \in \mathcal{N}} \delta^2 \left( \bar{\pi}_s + \sum_{i \in \mathcal{P}} \bar{\pi}_i c_{is} \right), \end{aligned} \quad (2.15)$$

and we have similar formulae to the above but with  $[I, J]^2$  replacing  $\langle R, S \rangle^2$ .

### 3 The amplitude as an integral over constraints

In order to take into account the constraints  $\mathcal{C}_K$  in evaluating the amplitude (2.14), we will rewrite the world sheet integration on  $\rho$  and  $k$  as integrals over a set of independent link variables.

In (2.14), we have

$$c_{js} = \frac{\kappa_j}{\kappa_s} \prod_{\substack{r \neq s \\ r \in \mathcal{N}}} \frac{\rho_j - \rho_r}{\rho_s - \rho_r}, \quad j \in \mathcal{P}, \quad s \in \mathcal{N}, \quad (3.1)$$

but it is convenient to change variables, defining

$$k_j = \prod_{r \in \mathcal{N}} (\rho_j - \rho_r) \kappa_j, \quad k_s = \prod_{\substack{r \neq s \\ r \in \mathcal{N}}} (\rho_s - \rho_r) \kappa_s, \quad j \in \mathcal{P}, \quad s \in \mathcal{N},$$

so that

$$c_{js} = \frac{k_j}{k_s} \frac{1}{\rho_j - \rho_s}. \quad (3.2)$$

Then (2.14) is left unchanged if we replace  $\kappa_\alpha$  by  $k_\alpha$ , and the action of the invariance group  $GL(2, \mathbb{R})$  is now given by

$$\rho_\alpha \mapsto \frac{a\rho_\alpha + b}{c\rho_\alpha + d}, \quad k_j \mapsto (ad - bc)k_j / (c\rho_j + d), \quad k_r \mapsto k_r(c\rho_r + d), \quad \alpha \in \mathcal{A}, \quad j \in \mathcal{P}, \quad s \in \mathcal{N},$$

$\mathcal{A} = \mathcal{P} \cup \mathcal{N}$ . Writing

$$f_{ir}(c) = \langle i, r \rangle - \sum_{s \in \mathcal{N}} c_{is} \langle s, r \rangle, \quad f_{rt}(c) = [r, t] + \sum_{i \in \mathcal{P}} [r, i] c_{it}, \quad r = R, S, \quad i \in \mathcal{P}, \quad t \in \mathcal{N}', \quad (3.3)$$

in the expression (2.14) we have

$$\begin{aligned} & \langle R, S \rangle^2 \prod_{i \in \mathcal{P}} \delta^2 \left( \pi_i - \sum_{r \in \mathcal{N}} c_{ir} \pi_r \right) \prod_{t \in \mathcal{N}'} \delta^2 \left( \bar{\pi}_t + \sum_{i \in \mathcal{P}} \bar{\pi}_i c_{it} \right) \\ &= K_1 \prod_{\substack{i \in \mathcal{P} \\ r=R, S}} \delta^2(f_{ir}(c)) \prod_{\substack{t \in \mathcal{N}' \\ r=R, S}} \delta^2(f_{rt}(c)) \equiv K_1 \delta^{N'}(f(c)), \end{aligned} \quad (3.4)$$

where  $K_1 = \langle R, S \rangle^{m+2} [R, S]^{n-2}$ .

Use  $\varrho_\ell, 1 \leq \ell \leq N'$ , to denote generically  $\{\rho_\alpha, k_\alpha : 1 \leq \alpha \leq N, \alpha \neq R, S\}$  and divide the  $mn$  variables  $c_{ir}$  into two subsets:  $c'_\ell, 1 \leq \ell \leq N'$ , and  $c''_K, 1 \leq K \leq N_R$ , (e.g., fix  $I, J \in \mathcal{P}$ , in addition to  $R, S \in \mathcal{N}$ , and take the  $c'$  subset to consist of  $\{c_{iR}, c_{iS}, c_{Ir}, c_{Jr} : i \in \mathcal{P}, r \in \mathcal{N}'\}$ , and the  $c''$  subset to consist of  $\{c_{ir} : i \neq I, J, r \neq R, S\}$ ). Then (2.14) can be written

$$\mathcal{M}^{\epsilon_1 \dots \epsilon_N} = K_1 \int \Psi(\varrho) \delta^{N'}(f) d^{N'} \varrho \quad (3.5)$$

where  $f \equiv \tilde{f}(\varrho) = f(c(\varrho))$  and

$$\Psi(\varrho) = (\rho_R - \rho_S)^2 \prod_{\alpha} \frac{1}{\rho_\alpha - \rho_{\alpha+1}} \prod_{\alpha \neq R, S} \frac{1}{k_\alpha}. \quad (3.6)$$



In principle, we could use the  $N'$  delta functions to perform the  $N'$  integrals over  $\varrho_\ell$  and then calculate the amplitude as a function of the momenta by solving the equations (3.3) to give  $\rho$  and  $k$  in terms of the momenta and then substitute for them in

$$\mathcal{M}^{\epsilon_1 \dots \epsilon_N} = K_1 \Psi(\varrho) \left| \frac{\partial(f)}{\partial(\varrho)} \right|^{-1}, \quad (3.7)$$

but this is not a computationally convenient way forward.

Instead, we seek to rewrite (3.5) first as an integral over all the  $mn$  variables  $c_{ir}$  and then to use the delta functions  $\delta^{N'}(f(c))$  to perform  $N'$  of these integrations to leave an integral over  $N_R$  variables corresponding to the constraints  $\mathcal{C}_K$ . We can use these  $N_R$  constraints to express the  $N_R$  variables  $c''_K$  as functions of the remaining  $N'$  variables  $c'_\ell$ ,  $c'' = \hat{c}''(c')$  and thus obtain  $N'$  functions  $\hat{f}(c') = f(c', \hat{c}''(c'))$ ; these are the functions we would obtain if we used the  $N'$  equations (3.2), corresponding to the  $c'$  to express  $\rho, k$  as functions of the  $c'$ .

$$\begin{aligned} \int \Psi(\varrho) \delta^{(N')}(\tilde{f}) d^{N'} \varrho &= \int \Psi(\varrho) \delta^{N'}(\hat{f}) \left| \frac{\partial(c')}{\partial(\varrho)} \right|^{-1} d^{N'} c' \\ &= \int \Psi(\varrho) \left| \frac{\partial(c')}{\partial(\varrho)} \right|^{-1} \delta^{N'}(\hat{f}) \delta^{N_R}(\mathcal{C}) \left| \frac{\partial(\mathcal{C})}{\partial(c'')} \right| d^{mn} c \\ &= \int \Psi(\varrho) \left| \frac{\partial(c')}{\partial(\varrho)} \right|^{-1} \left| \frac{\partial(\mathcal{C})}{\partial(c'')} \right| \delta^{N'}(f) \delta^{N_R}(\mathcal{C}) d^{mn} c \\ &= \int \Psi(\varrho) \left| \frac{\partial(c')}{\partial(\varrho)} \right|^{-1} \left| \frac{\partial(\mathcal{C})}{\partial(c'')} \right| \delta^{N'}(f) \delta^{N_R}(\hat{\mathcal{C}}) d^{mn} c \\ &= \int \Psi(\varrho) \left| \frac{\partial(c')}{\partial(\varrho)} \right|^{-1} \left| \frac{\partial(\mathcal{C})}{\partial(c'')} \right| \left| \frac{\partial(f)}{\partial(c')} \right|^{-1} \delta^{N_R}(\hat{\mathcal{C}}) d^{N_R} c'' \end{aligned} \quad (3.8)$$

with  $\hat{\mathcal{C}}(c'') = \mathcal{C}(\hat{c}'(c''), c'')$ , where the functions  $c' = \hat{c}'(c'')$  are obtained by using the  $N'$  equations  $f(c', c'') = 0$  to express  $c'$  in terms of  $c''$ . The Jacobian of  $f$  with respect to  $c'$  is a constant, dependent only on momenta rather than the  $c_{ir}$ , because the  $f$  are linear; the value of the constant depends on the choice of the  $c'_\ell$ . If we use the choice  $\{c_{iR}, c_{iS}, c_{Ir}, c_{Jr} : i \in \mathcal{P}, r \in \mathcal{N}'\}$  for  $c'$ ,

$$\left| \frac{\partial(f)}{\partial(c')} \right| = \langle R, S \rangle^{2m} [R, S]^{n-2} [I, J]^{n-2}.$$

Then

$$\mathcal{M}^{\epsilon_1 \dots \epsilon_N} = K \int F(c'') \delta^{N_R}(\hat{\mathcal{C}}) d^{N_R} c'' \quad (3.9)$$

with

$$F(c'') = \Psi(\varrho) \left| \frac{\partial(c')}{\partial(\varrho)} \right|^{-1} \left| \frac{\partial(\mathcal{C})}{\partial(c'')} \right| \quad (3.10)$$

and

$$K = K_1 \left| \frac{\partial(f)}{\partial(c')} \right|^{-1} = \langle R, S \rangle^{2-m} [I, J]^{2-n}.$$

We can integrate (3.9) to obtain

$$\mathcal{M}^{\epsilon_1 \dots \epsilon_N} = K \sum F(c'') \left| \frac{\partial(\hat{\mathcal{C}})}{\partial(c'')} \right|^{-1}, \quad (3.11)$$

where the sum is over at least some of the simultaneous solutions of the  $N_R$  constraint equations  $\hat{C}_K(c'') \equiv C_K(\check{c}'(c''), c'') = 0$ . [Note that in (3.11) the Jacobian is calculated for  $\hat{\mathcal{C}}$ , that is for  $\mathcal{C}$  regarded as a function of  $c''$  with  $c'$  put equal to  $\check{c}'(c'')$ , whereas in (3.10) is for  $\mathcal{C}$  with respect to  $c''$ , with all the  $c$  regarded as independent.] To find a rational answer for the amplitude, in line with the known results, we need to sum over all the solutions  $c''$  with appropriate signs or phases that enable the contributions to be combined into a contour integral of the form

$$\mathcal{M}^{\epsilon_1 \dots \epsilon_N} = K \oint F(c'') \prod_K \frac{dc''_K}{\hat{C}_K}. \quad (3.12)$$

[Here we understand the notation for the contour integral to include appropriate factors of  $2\pi i$ .] This will become apparent when we discuss the 6-point function, with  $m = n = 3$ , in detail in section 6. But first we will discuss the form of the integrand  $F(c)$  in section 4, and the parameterization of the general solution for  $c_{ir}$  of the linear equations  $f_\ell = 0$  in section 5.

#### 4 The form of the integrand, $F(c)$

We can now give a general prescription for the integrand  $F(c)$  in terms of the link variables, working from the string function (3.6). As before, we fix  $I, J \in \mathcal{P}$  and  $R, S \in \mathcal{N}$ , set  $\mathcal{P}' = \{k \in \mathcal{P} : k \neq i, j\}$  and  $\mathcal{N}' = \{t \in \mathcal{N} : t \neq r, s\}$ , and chose for the  $N_R$  variables  $c''$  the collection  $\{c_{kt} : k \in \mathcal{P}', t \in \mathcal{N}'\}$ . Then the remaining variables  $c'$  are  $\{c_{iR}, c_{iS}, c_{It}, c_{Jt} : i \in \mathcal{P}, t \in \mathcal{N}'\}$ . Correspondingly, we take the  $N_R$  constraint functions  $\mathcal{C}_K$  to be

$$\mathcal{C}_{kt} \equiv \mathcal{C}_{RSt}^{IJk}, \quad k \in \mathcal{P}', t \in \mathcal{N}'. \quad (4.1)$$

If  $k, k' \in \mathcal{P}'$ , and  $t, t' \in \mathcal{N}'$ ,

$$\frac{\partial \mathcal{C}_{kt}}{\partial c_{k't'}} = 0 \quad \text{unless } k = k' \text{ and } t = t', \quad (4.2)$$

so that

$$\begin{aligned} \left| \frac{\partial(\mathcal{C})}{\partial(c'')} \right| &= \prod_{\substack{k \in \mathcal{P}' \\ t \in \mathcal{N}'}} \frac{\partial \mathcal{C}_{kt}}{\partial c_{kt}}. \\ &= [c_{iR}c_{jS} - c_{iS}c_{jR}]^{N_R} \prod_{t \in \mathcal{N}'} [c_{It}c_{Jt}]^{m-2} \prod_{k \in \mathcal{P}'} [c_{kR}c_{kS}]^{n-2} \prod_{\substack{k \in \mathcal{P}' \\ t \in \mathcal{N}'}} \frac{1}{c_{kt}}, \end{aligned} \quad (4.3)$$

using the expression we find from (1.7),

$$\begin{aligned} \frac{\partial \mathcal{C}_{kt}}{\partial c_{kt}} &= c_{kS}c_{iR}c_{jR}(c_{It}c_{jS} - c_{iS}c_{Jt}) - c_{kR}c_{iS}c_{jS}(c_{It}c_{jR} - c_{iR}c_{Jt}) \\ &= c_{It}c_{Jt}c_{kR}c_{kS}(c_{iR}c_{jS} - c_{iS}c_{jR})/c_{kt}. \end{aligned} \quad (4.4)$$

Also it is not difficult to see that the Jacobian of  $c'$  with respect to  $\rho, k$  can be written as a product of factors:

$$\begin{aligned}
 \left| \frac{\partial(c')}{\partial(\rho, k)} \right| &= \left| \frac{\partial(c_{IR}, c_{IS})}{\partial(\rho_I, k_I)} \right| \times \left| \frac{\partial(c_{JR}, c_{JS})}{\partial(\rho_J, k_J)} \right| \times \prod_{k \in \mathcal{P}'} \left| \frac{\partial(c_{kR}, c_{kS})}{\partial(\rho_k, k_k)} \right| \times \prod_{t \in \mathcal{N}'} \left| \frac{\partial(c_{It}, c_{Jt})}{\partial(\rho_t, k_t)} \right| \\
 &= \frac{k_I^{n-1} k_J^{n-1}}{k_R^m k_S^m} (\rho_R - \rho_S)^m (\rho_I - \rho_J)^{n-2} \prod_{k \in \mathcal{P}'} k_k \times \prod_{t \in \mathcal{N}'} \frac{1}{k_t^3} \\
 &\quad \times \prod_{l \in \mathcal{P}} \frac{1}{(\rho_l - \rho_R)^2 (\rho_l - \rho_S)^2} \prod_{t \in \mathcal{N}'} \frac{1}{(\rho_I - \rho_t)^2 (\rho_J - \rho_t)^2}, \tag{4.5}
 \end{aligned}$$

using

$$\begin{aligned}
 \frac{\partial(c_{IR}, c_{IS})}{\partial(\rho_I, k_I)} &= -\frac{k_I}{k_R k_S} \frac{(\rho_R - \rho_S)}{(\rho_I - \rho_R)^2 (\rho_I - \rho_S)^2}, \\
 \frac{\partial(c_{It}, c_{Jt})}{\partial(\rho_t, k_t)} &= \frac{k_I k_J}{(k_t)^3} \frac{(\rho_I - \rho_J)}{(\rho_I - \rho_t)^2 (\rho_J - \rho_t)^2}.
 \end{aligned}$$

So

$$\begin{aligned}
 (\rho_R - \rho_S)^2 \left| \frac{\partial(c')}{\partial(\rho, k)} \right|^{-1} &\prod_{l \in \mathcal{P}} \frac{1}{k_l} \prod_{k \in \mathcal{N}'} \frac{1}{k_k} \tag{4.6} \\
 &= \frac{k_R^{2-m} k_S^{2-m}}{k_I^{2-n} k_J^{2-n}} (\rho_R - \rho_S)^{2-m} (\rho_I - \rho_J)^{2-n} c_{IR}^{-2} c_{IS}^{-2} c_{JR}^{-2} c_{JS}^{-2} \prod_{k \in \mathcal{P}'} c_{kR}^{-2} c_{kS}^{-2} \prod_{t \in \mathcal{N}'} c_{It}^{-2} c_{Jt}^{-2} \prod_{l \in \mathcal{P}} k_l^2 \prod_{u \in \mathcal{N}} \frac{1}{k_u^2},
 \end{aligned}$$

and, using

$$\frac{k_r k_s}{k_i k_j} (\rho_i - \rho_j) (\rho_s - \rho_r) = \frac{c_{ir} c_{js} - c_{is} c_{jr}}{c_{ir} c_{is} c_{jr} c_{js}}, \quad \text{for } i, j \in \mathcal{P}, r, s \in \mathcal{N}, \tag{4.7}$$

we have

$$\begin{aligned}
 (\rho_R - \rho_S)^2 \left| \frac{\partial(c')}{\partial(\rho, k)} \right|^{-1} \left| \frac{\partial(\mathcal{C})}{\partial(c'')} \right| &\prod_{i \in \mathcal{P}} \frac{1}{k_i} \prod_{k \in \mathcal{N}'} \frac{1}{k_k} \\
 &= \frac{k_I^n k_J^n}{k_R^m k_S^m} (\rho_R - \rho_S)^{-m} (\rho_I - \rho_J)^{-n} \frac{[c_{IR} c_{JS} - c_{IS} c_{JR}]^{N_R+2}}{c_{IR}^3 c_{IS}^3 c_{JR}^3 c_{JS}^3} \\
 &\quad \times \prod_{t \in \mathcal{N}'} c_{It}^{m-3} c_{Jt}^{m-3} \prod_{k \in \mathcal{P}'} c_{kR}^{n-3} c_{kS}^{n-3} \prod_{\substack{k \in \mathcal{P} \\ t \in \mathcal{N}}} \frac{1}{c_{kt}} \prod_{l \in \mathcal{P}} k_l^2 \prod_{u \in \mathcal{N}} \frac{1}{k_u^2}, \tag{4.8}
 \end{aligned}$$

and finally

$$\begin{aligned}
 F(c) &= \Psi(\rho, k) \left| \frac{\partial(c')}{\partial(\rho, k)} \right|^{-1} \left| \frac{\partial(\mathcal{C})}{\partial(c'')} \right| \\
 &= \frac{k_I^n k_J^n}{k_R^m k_S^m} (\rho_R - \rho_S)^{-m} (\rho_I - \rho_J)^{-n} \frac{[c_{IR} c_{JS} - c_{IS} c_{JR}]^{N_R+2}}{c_{IR}^3 c_{IS}^3 c_{JR}^3 c_{JS}^3} \prod_{t \in \mathcal{N}'} c_{It}^{m-3} c_{Jt}^{m-3} \prod_{k \in \mathcal{P}'} c_{kR}^{n-3} c_{kS}^{n-3} \\
 &\quad \times \prod_{\substack{k \in \mathcal{P} \\ t \in \mathcal{N}}} \frac{1}{c_{kt}} \prod_{l \in \mathcal{P}} k_l^2 \prod_{u \in \mathcal{N}} \frac{1}{k_u^2} \prod_{\alpha=1}^N \frac{1}{\rho_\alpha - \rho_{\alpha+1}}. \tag{4.9}
 \end{aligned}$$

We can write this final product in terms of the  $c_{ir}$  by using (3.2) whenever  $\epsilon_\alpha = -\epsilon_{\alpha+1}$ ; and using (4.7), with a factor of  $(\rho_R - \rho_S)$  supplied when  $\epsilon_\alpha = \epsilon_{\alpha+1} = +$ , and a factor of  $(\rho_I - \rho_J)$  supplied when  $\epsilon_\alpha = \epsilon_{\alpha+1} = -$ . This will leave  $(\rho_I - \rho_J)^{-p}(\rho_R - \rho_S)^{-p}$ , where  $p$  is the number of sign changes going from  $\epsilon_1$  to  $\epsilon_N$  and back to  $\epsilon_1$ . These factors can again be converted using (4.7), yielding a rational expression for  $F(c)$  of order  $6(m-2)(n-2) - mn$ . We now give some examples and a general prescription.

- (a) For  $m = n$  and  $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \dots, \epsilon_{2n-1}, \epsilon_{2n}) = (+, -, +, -, \dots, +, -)$ ,

$$F(c) = [c_{IR}c_{JS} - c_{IS}c_{JR}]^{(n-2)(n-3)} \prod_{t \in \mathcal{N}'} c_{It}^{n-3} c_{Jt}^{n-3} \prod_{k \in \mathcal{P}} c_{kR}^{n-3} c_{kS}^{n-3} \prod'_{\substack{k \in \mathcal{P} \\ t \in \mathcal{N}}} \frac{1}{c_{kt}}, \quad (4.10)$$

where the prime on the last product indicates that terms  $1/c_{kt}$  should be omitted when  $k, t$  are adjacent.

- (b) For  $(\epsilon_1, \dots, \epsilon_m, \epsilon_{m+1}, \dots, \epsilon_{m+n}) = (+, \dots, +, -, \dots, -)$ , where we choose the labeling  $\epsilon_i = +$ ,  $1 \leq i \leq m$ ;  $\epsilon_r = -$ ,  $m+1 \leq r \leq N$ ; and  $I = 1, J = m, R = m+1, S = N$ , we have

$$F(c) = -c_{IS}c_{JR} [c_{IR}c_{JS} - c_{IS}c_{JR}]^{N_{R+1}} \prod_{i=1}^{m-1} \frac{1}{[c_{iR}c_{i+1,S} - c_{i+1,R}c_{iS}]} \quad (4.11)$$

$$\times \prod_{r=m+1}^{N-1} \frac{1}{[c_{Ir}c_{J,r+1} - c_{I,r+1}c_{Jr}]} \prod_{t \in \mathcal{N}'} c_{It}^{m-1} c_{Jt}^{m-1} \prod_{k \in \mathcal{P}'} c_{kR}^{n-1} c_{kS}^{n-1} \prod'_{\substack{k \in \mathcal{P} \\ t \in \mathcal{N}}} \frac{1}{c_{kt}}.$$

- (c) In general, if  $(\epsilon_1, \dots, \epsilon_N)$ , begins with  $\epsilon_1 = +$  and ends with  $\epsilon_N = -$  and comprises  $p$  strings with  $\epsilon_\alpha = +$  and, therefore,  $p$  strings with  $\epsilon_\alpha = -$ , then, up to sign,  $F(c)$  is given by

$$F(c) = [c_{IR}c_{JS} - c_{IS}c_{JR}]^{N_{R-p+2}} c_{IR}^{p-3} c_{IS}^{p-3} c_{JR}^{p-3} c_{JS}^{p-3} \prod_{t \in \mathcal{N}'} c_{It}^{m-3} c_{Jt}^{m-3}$$

$$\times \prod_{k \in \mathcal{P}'} c_{kR}^{n-3} c_{kS}^{n-3} \prod_{\substack{k \in \mathcal{P} \\ t \in \mathcal{N}}} \frac{1}{c_{kt}} \prod_{\alpha=1}^N d_{\alpha, \alpha+1}, \quad (4.12)$$

where

$$d_{ir} = c_{ir}, \quad d_{ri} = c_{ir}, \quad d_{ij} = \frac{c_{iR}c_{jS}c_{jR}c_{iS}}{c_{iR}c_{jS} - c_{jR}c_{iS}}, \quad d_{rs} = \frac{c_{Ir}c_{Js}c_{Is}c_{Jr}}{c_{Ir}c_{Js} - c_{Is}c_{Jr}},$$

$$i, j \in \mathcal{P}, \quad r, s \in \mathcal{N}.$$

Note, one may obtain different expressions for the integrands using the identity

$$\frac{c_{iR}c_{jS} - c_{jR}c_{iS}}{c_{IR}c_{JS} - c_{IS}c_{JR}} = \frac{c_{ir}c_{js} - c_{is}c_{jr}}{c_{Ir}c_{Js} - c_{Is}c_{Jr}} \times \frac{c_{iR}c_{jS}c_{jR}c_{iS}c_{Ir}c_{Js}c_{Is}c_{Jr}}{c_{ir}c_{js}c_{is}c_{jr}c_{IR}c_{JS}c_{IS}c_{JR}}.$$

For given  $m$  and  $n$ , the expressions for  $F(c)$  for different orderings of the helicities are related by the transformations given in appendix A.

## 5 Parameterization of the general solution of linear constraints on $c_{ir}$

In this section we will parameterize the general solution to the linear constraints (1.1), (1.2) on the link variables in order to express them in terms of suitable independent variables over which to perform the multi-dimensional contour integral to obtain the amplitudes. As remarked in section 1, the  $2N$  linear equations (1.1), (1.2) imply energy-momentum conservation. Thus they typically provide  $N' = 2N - 4$  constraints on the variables  $c_{ir}$ , leaving  $N_R = mn - N'$  degrees of freedom. These remaining degrees of freedom are determined by the  $N_R$  independent constraints  $\mathcal{C}_K$  that follow from the requirement that  $c_{ir}$  be of the form (3.2), and we shall discuss the form of these non-linear constraints in this section.

A solution to the linear equations (1.1), (1.2) is always provided by  $c_{ir} = a_{ir}$ , where

$$a_{ir} = \frac{1}{p^2} \sum_{j \in \mathcal{P}} \langle i, j \rangle [j, r] = -\frac{1}{p^2} \sum_{s \in \mathcal{N}} \langle i, s \rangle [s, r], \quad (5.1)$$

using energy-momentum conservation, and

$$p = \sum_{j \in \mathcal{P}} p_j = -\sum_{r \in \mathcal{N}} p_r, \quad p^2 = \sum_{\substack{i < j \\ i, j \in \mathcal{P}}} \langle i, j \rangle [i, j] = \sum_{\substack{r < s \\ r, s \in \mathcal{N}}} \langle r, s \rangle [r, s]. \quad (5.2)$$

To show that  $c_{ir} = a_{ir}$  satisfies (1.1), (1.2), first note that

$$\langle i, j \rangle \pi_k + \langle j, k \rangle \pi_i + \langle k, i \rangle \pi_j = 0, \quad \text{for any } i, j, k,$$

because, taking the angle bracket with any vector  $\pi_l$ ,

$$\langle i, j \rangle \langle k, l \rangle + \langle j, k \rangle \langle i, l \rangle = \langle k, l \rangle \langle i, j \rangle - \langle k, j \rangle \langle i, l \rangle = -\langle k, i \rangle \langle j, l \rangle.$$

Similarly

$$[r, s] \bar{\pi}_t + [s, t] \bar{\pi}_r + [t, r] \bar{\pi}_s = 0, \quad \text{for any } r, s, t. \quad (5.3)$$

So

$$\begin{aligned} \sum_{r \in \mathcal{N}} a_{ir} \pi_r &= -\frac{1}{p^2} \sum_{r, s \in \mathcal{N}} \langle i, s \rangle [s, r] \pi_r \\ &= -\frac{1}{2p^2} \sum_{r, s \in \mathcal{N}} [s, r] (\langle i, s \rangle \pi_r - \langle i, r \rangle \pi_s) \\ &= \frac{1}{2p^2} \sum_{r, s \in \mathcal{N}} [s, r] \langle s, r \rangle \pi_i = \pi_i, \end{aligned} \quad (5.4)$$

establishing (1.1), (1.2) follows similarly.

For convenience write

$$A_{ir} = p^2 a_{ir}. \quad (5.5)$$

Then, for  $i, j \in \mathcal{P}$ ,  $r, s \in \mathcal{N}$ ,

$$\begin{aligned}
 A_{ir}A_{js} - A_{is}A_{jr} &= \sum_{u,v \in \mathcal{N}} (\langle i, u \rangle [u, r] \langle j, v \rangle [v, s] - \langle i, u \rangle [u, s] \langle j, v \rangle [v, r]) \\
 &= \sum_{u,v \in \mathcal{N}} \langle i, u \rangle \langle j, v \rangle ([u, r][v, s] - [u, s][v, r]) \\
 &= [r, s] \sum_{u,v \in \mathcal{N}} \langle i, u \rangle \langle j, v \rangle [u, v] = [r, s] \sum_{\substack{u < v \\ u,v \in \mathcal{N}}} (\langle i, u \rangle \langle j, v \rangle - \langle i, v \rangle \langle j, u \rangle) [u, v] \\
 &= p^2 [r, s] \langle i, j \rangle
 \end{aligned} \tag{5.6}$$

and so, for  $i, j, k \in \mathcal{P}$ ,  $r, s, t \in \mathcal{N}$ ,

$$\begin{aligned}
 (p^2)^3 \begin{vmatrix} a_{ir} & a_{is} & a_{it} \\ a_{jr} & a_{js} & a_{jt} \\ a_{kr} & a_{ks} & a_{kt} \end{vmatrix} &= (A_{ir}A_{js} - A_{jr}A_{is})A_{kt} + (A_{jr}A_{ks} - A_{kr}A_{js})A_{it} + (A_{kr}A_{is} - A_{ir}A_{ks})A_{jt} \\
 &= p^2 [r, s] (\langle i, j \rangle A_{kt} + \langle j, k \rangle A_{it} + \langle k, i \rangle A_{jt}) = 0,
 \end{aligned} \tag{5.7}$$

using (5.2).

Since this determinant vanishes for any  $i, j, k \in \mathcal{P}$ ,  $r, s, t \in \mathcal{N}$ , this implies that the matrix

$$\begin{pmatrix} a_{i_1 r_1} & a_{i_1 r_2} & \cdots & a_{i_1 r_n} \\ a_{i_2 r_1} & a_{i_2 r_2} & \cdots & a_{i_2 r_n} \\ \vdots & \vdots & & \vdots \\ a_{i_m r_1} & a_{i_m r_2} & \cdots & a_{i_m r_n} \end{pmatrix} \tag{5.8}$$

has rank 2. In fact, this is evident from the fact that  $a_{ir}$  is defined in (5.1) as the product of the  $m \times m$  dimensional matrix  $\langle i, j \rangle$  and the  $m \times n$  dimensional matrix  $[j, r]$ ; because the  $\pi_i$  are two-dimensional, the matrix  $\langle i, j \rangle$  has rank at most two and the rank of  $a_{ir}$  can not be larger. For  $m, n > 2$ , this condition is different from the rather more unusual condition (1.6).

The condition that the matrix (1.6) have rank 2 is sufficient as well as necessary for  $c_{jr}$  to be of the form

$$c_{jr} = \frac{k_j}{k_r(\rho_j - \rho_r)}.$$

We can prove this by induction on the size of (1.6). Suppose (1.6) is of size  $m \times n$ , has rank 2 and that the result holds for matrices of this size; consider adding an additional column  $x_i = (c_{i,n+1})^{-1}$ ,  $1 \leq i \leq m$ , while leaving the rank of the matrix at 2. Then

$$x_i = \lambda \frac{k_1}{k_i} (\rho_i - \rho_1) + \mu \frac{k_2}{k_i} (\rho_i - \rho_2) = \frac{k_{n+1}}{k_i} (\rho_i - \rho_{n+1}), \tag{5.9}$$

where

$$k_{n+1} = \lambda k_1 + \mu k_2, \quad \rho_{n+1} = \frac{\lambda k_1 \rho_1 + \mu k_2 \rho_2}{\lambda k_1 + \mu k_2},$$

so that the elements of the additional column are

$$c_{i,n+1} = \frac{k_i}{k_{n+1}(\rho_i - \rho_{n+1})},$$

which is of the required form.

For  $m = 2$  or  $n = 2$ , (1.1), (1.2) determine  $c_{ir}$  uniquely, and it is straightforward then to check that  $c_{ir} = a_{ir}$  provides the well known MHV amplitudes. For  $n = 2$ ,  $\mathcal{N} = \{R, S\}$ ,

$$c_{iR} = -\frac{1}{p^2} \langle i, S \rangle [S, R] = \frac{\langle i, S \rangle}{\langle R, S \rangle}, \quad c_{iS} = \frac{\langle i, R \rangle}{\langle S, R \rangle}, \quad \text{as } p^2 = \langle R, S \rangle [R, S]$$

in this case. Then

$$c_{i,R}c_{i+1,S} - c_{i,S}c_{i+1,R} = \frac{1}{p^2} \langle i, i+1 \rangle [R, S] = \frac{\langle i, i+1 \rangle}{\langle R, S \rangle}.$$

Then, from the formulae at the end of section 4,

$$\mathcal{M}^{+\dots+--} = \langle R, S \rangle^4 \left/ \prod_{\alpha=1}^N \langle \alpha, \alpha+1 \rangle \right.,$$

since there are no integrations to perform in this case. Similarly for  $m = 2$ ,  $\mathcal{P} = \{I, J\}$ ,

$$c_{Ir} = \frac{1}{p^2} \langle I, J \rangle [J, r] = -\frac{[J, r]}{[J, I]}, \quad c_{Jr} = -\frac{[I, r]}{[I, J]}, \quad \text{as } p^2 = \langle I, J \rangle [I, J]$$

in this case, leading to the familiar form of the amplitude for this case.

For  $m, n > 2$  we must add to  $a_{ir}$  a solution of the corresponding homogeneous linear equations,

$$\sum_{r \in \mathcal{N}} \hat{c}_{jr} \pi_r = 0, \quad j \in \mathcal{P} \tag{5.10}$$

$$\sum_{i \in \mathcal{P}} \bar{\pi}_i \hat{c}_{is} = 0, \quad s \in \mathcal{N}, \tag{5.11}$$

in order to obtain a solution of (1.1), (1.2) that has the property that (1.6) is of rank 2, when

$$c_{ir} = a_{ir} + \hat{c}_{ir}. \tag{5.12}$$

The  $\hat{c}_{ir}$  lie in an  $N_R$ -dimensional space. For  $m = n = 3$ ,  $\mathcal{P} = \{i, j, k\}$ ,  $\mathcal{N} = \{r, s, t\}$ , this space is one-dimensional and parametrized by

$$\hat{c}_{ir} = \frac{\beta}{4p^2} \epsilon_{ijk} [j, k] \epsilon_{rst} \langle s, t \rangle.$$

The single constraint from (1.6),  $\mathcal{C}_1 \equiv \mathcal{C}_{rst}^{ijk} = 0$  provides a quartic equation to determine  $\beta$ , which we shall discuss further in section 6.

For general  $m, n \geq 3$ , the general solution to (5.10), (5.11) is provided by

$$\hat{c}_{ir} = \frac{1}{4} \sum_{\substack{j, k \in \mathcal{P} \\ s, t \in \mathcal{N}}} \beta_{rst}^{ijk} [j, k] \langle s, t \rangle,$$

where  $\beta_{rst}^{ijk}$  is antisymmetric under permutations of  $i, j, k$  and also under the permutations of  $r, s, t$ ; it follows from (5.2), (5.3) that this satisfies (5.10), (5.11). Because there are only  $(m-2)(n-2)$  independent solutions to (5.10), (5.11) there is some arbitrariness in the choice of  $\beta_{rst}^{ijk}$  for a given solution  $c_{ir}$ , e.g.

$$\beta_{rst}^{ijk} \mapsto \beta_{rst}^{ijk} + [i, l]\gamma_{rst}^{jk} + [j, l]\gamma_{rst}^{ki} + [k, l]\gamma_{rst}^{ij},$$

where  $\gamma_{rst}^{ij}$  is antisymmetric in  $i, j$  and in  $r, s, t$ , and  $\bar{\pi}_l$  is arbitrary, leaves the solution  $\hat{c}_{ir}$  unchanged.

We can express each of the  $\hat{c}_{ir}$  as a linear combination of the  $N_R$  components

$$\hat{c}_{kt} = \beta_{kt}[I, J]\langle R, S \rangle, \quad k \in \mathcal{P}', t \in \mathcal{N}'$$

by

$$\begin{aligned} \hat{c}_{It} &= \sum_{k \in \mathcal{P}'} \beta_{kt}[J, k]\langle R, S \rangle, & \hat{c}_{kR} &= \sum_{t \in \mathcal{N}'} \beta_{kt}[I, J]\langle S, t \rangle, & \hat{c}_{IR} &= \sum_{\substack{k \in \mathcal{P}' \\ t \in \mathcal{N}'}} \beta_{kt}[J, k]\langle S, t \rangle, \\ & k \in \mathcal{P}', t \in \mathcal{N}', \end{aligned} \tag{5.13}$$

and similar expressions for  $\hat{c}_{kS}, \hat{c}_{Jt}, \hat{c}_{IS}, \hat{c}_{JR}, \hat{c}_{JS}$ . This is equivalent to taking  $\beta_{rst}^{ijk} = 0$  unless exactly one of  $i, j, k$  is in  $\mathcal{P}'$  and exactly one of  $r, s, t$  is in  $\mathcal{N}'$ . In this case, the only possibly nonzero components of  $\beta_{rst}^{ijk}$  are

$$\beta_{RSt}^{IJk} = \beta_{kt}, \quad k \in \mathcal{P}', t \in \mathcal{N}' \tag{5.14}$$

and those related to these by antisymmetry under permutations of  $I, J, k$  or of  $R, S, t$ .

$$\beta_{rst}^{ijk} = \epsilon_{ijk}\epsilon_{rst}\beta, \quad \text{for } m = n = 3, \tag{5.15}$$

and

$$\beta_{rst}^{ijk} = \beta^{ijk}\epsilon_{rst}, \quad \text{for } m > 3, n = 3, \tag{5.16}$$

and, in this case, we can specify that the only nonzero components of  $\beta^{ijk}$  are the  $m-2$  components of the form  $\beta^{IJk} = \beta_k, k \in \mathcal{P}'$  and those obtained by permuting  $I, J, k$ .

Then, from (5.12), we have,

$$c_{ir} = a_{ir} + \frac{1}{4p^2} \sum_{\substack{j, k \in \mathcal{P}' \\ s, t \in \mathcal{N}'}} \beta_{rst}^{ijk}[j, k]\langle s, t \rangle, \tag{5.17}$$

where the parameters  $\beta_{rst}^{ijk}$  are zero unless they are related to

$$\beta_{RSt}^{IJk} = \beta_k \quad k \in \mathcal{P}', t \in \mathcal{N}',$$

by permutation of  $i, j, k$  and  $r, s, t$ . Then, from (3.12), the amplitude becomes

$$\mathcal{M}^{\epsilon_1 \dots \epsilon_N} = \tilde{K} \oint F(c) \prod_{\substack{k \in \mathcal{P}' \\ t \in \mathcal{N}'}} \frac{d\beta_{kt}}{\mathcal{C}_{kt}(\beta)}, \tag{5.18}$$

with

$$\tilde{K} = [I, J]^{(m-3)(n-2)} \langle R, S \rangle^{(m-2)(n-3)} (p^2)^{(m-2)(2-n)}, \tag{5.19}$$

and  $\mathcal{C}_{kt} = \mathcal{C}_{RSt}^{IJk}$  provide the  $N_R$  constraints  $\mathcal{C}_K$ .



## 6 NMHV 6-point functions

We will exemplify our analysis by computing all NMHV 6-point gluon tree amplitudes. Following [2], we then find equivalent but different expressions [24, 25] for the amplitudes, by choosing other equivalent contours and integrands.

For  $m = n = 3$ , we take  $\mathcal{P} = \{i, j, k\}, \mathcal{N} = \{r, s, t\}$ ; from (5.18),

$$\mathcal{M}^{\epsilon_1 \dots \epsilon_6} = \frac{1}{p^2} \oint F(c) \frac{d\beta}{\mathcal{C}(\beta)}, \quad (6.1)$$

with just one constraint  $\mathcal{C}_1 \equiv \mathcal{C}_{kt} \equiv \mathcal{C}(\beta)$ ,

$$\begin{aligned} \mathcal{C}(\beta) = & c_{ir}c_{js}c_{kt}c_{jr}c_{ks}c_{it} + c_{jr}c_{ks}c_{it}c_{kr}c_{is}c_{jt} + c_{kr}c_{is}c_{jt}c_{ir}c_{js}c_{kt} \\ & - c_{kr}c_{js}c_{it}c_{jr}c_{is}c_{kt} - c_{jr}c_{is}c_{kt}c_{ir}c_{ks}c_{jt} - c_{ir}c_{ks}c_{jt}c_{kr}c_{js}c_{it}; \end{aligned} \quad (6.2)$$

the link variables are

$$c_{ir} = a_{ir} + \frac{\beta}{p^2} [j, k] \langle s, t \rangle,$$

where  $(i, j, k)$  and  $(r, s, t)$  are cyclic, and

$$F(c) = \frac{k_i^2 k_j^2 k_k^2}{k_r^2 k_s^2 k_t^2} \prod_{\substack{k \in \mathcal{P} \\ t \in \mathcal{N}}} \frac{1}{c_{kt}} \prod_{\alpha=1}^6 \frac{1}{\rho_\alpha - \rho_{\alpha+1}}.$$

For the three different orderings of the NMHV 6-point functions, the integrands are given as in section 4.

(a) for the case  $(i, t, j, r, k, s) \equiv (+, -, +, -, +, -)$ ,

$$F^{+-+--+}(c) = -\frac{1}{c_{ir}c_{js}c_{kt}};$$

(b) for the case  $(k, i, s, t, j, r) \equiv (+, +, -, -, +, -)$

$$F^{++--+-}(c) = \frac{c_{is}}{(c_{ks}c_{it} - c_{kt}c_{is})c_{ir}c_{js}};$$

(c) for the case  $(i, j, k, r, s, t) \equiv (+, +, +, -, -, -)$

$$F^{+++---}(c) = -\frac{c_{js}}{(c_{ir}c_{js} - c_{is}c_{jr})(c_{js}c_{kt} - c_{jt}c_{ks})}.$$

To compute the amplitudes, we examine the explicit form of the constraint, which leads to a quartic equation for  $\beta$ . Writing

$$V_i = [j, k], \quad W_r = \langle s, t \rangle, \quad \bar{V}_i = \langle j, k \rangle, \quad \bar{W}_r = [s, t],$$

and similarly for cyclic rotations of  $(i, j, k)$  and of  $(r, s, t)$ , we have

$$c_{ir} = a_{ir} + \frac{\beta}{p^2} V_i W_r, \quad (6.3)$$

and, from (5.6),

$$A_{ir}A_{js} - A_{is}A_{jr} = p^2\bar{V}_k\bar{W}_t. \quad (6.4)$$

From appendix B, where we list some algebraic relations useful for computing the 6-point functions,

$$c_{ir}c_{js} - c_{is}c_{jr} = \beta\bar{c}_{tk}, \quad (6.5)$$

for

$$\bar{c}_{ri} = \bar{a}_{ri} + \frac{\bar{\beta}}{p^2}\bar{W}_r\bar{V}_i, \quad \bar{a}_{ri} = \frac{1}{p^2}\bar{A}_{ri}, \quad \bar{\beta} = 1/\beta, \quad (6.6)$$

$$\bar{A}_{ri} = \sum_{s \in \mathcal{N}} \langle r, s \rangle [s, i] = - \sum_{j \in \mathcal{P}} \langle r, j \rangle [j, i]. \quad (6.7)$$

In appendix B, we see that  $\det(c) = \beta$  so that  $\bar{c}_{ri}$  is the inverse of  $c_{ir}$ , and hence provides the general solution of

$$\pi_r = \sum_{i \in \mathcal{P}} \bar{c}_{ri}\pi_i, \quad \bar{\pi}_i = - \sum_{r \in \mathcal{N}} \bar{\pi}_r\bar{c}_{ri},$$

the equations for the amplitude with flipped helicities. Again, from appendix B,

$$\bar{A}_{ri}\bar{A}_{sj} - \bar{A}_{si}\bar{A}_{rj} = p^2W_tV_k,$$

and

$$\bar{c}_{ri}\bar{c}_{sj} - \bar{c}_{si}\bar{c}_{rj} = \bar{\beta}c_{kt}.$$

Using these properties of  $c_{ir}$ , the constraint becomes  $\mathcal{C}(\beta) = 0$ , where

$$\begin{aligned} \mathcal{C}(\beta) &= c_{is}c_{jr}(c_{kt}c_{ir} - c_{kr}c_{it})(c_{js}c_{kt} - c_{jt}c_{ks}) - c_{ir}c_{js}(c_{ks}c_{it} - c_{kt}c_{is})(c_{jt}c_{kr} - c_{jr}c_{kt}) \\ &= \beta^2[c_{is}c_{jr}\bar{c}_{ri}\bar{c}_{sj} - c_{ir}c_{js}\bar{c}_{si}\bar{c}_{rj}] \\ &= \beta c_{ir}c_{js}c_{kt} - \beta^3\bar{c}_{ri}\bar{c}_{sj}\bar{c}_{tk}, \end{aligned} \quad (6.8)$$

which is manifestly quartic in  $\beta$ . Writing

$$\mathcal{C}(\beta) = \beta c_{ir}c_{js}c_{kt} - \beta^3\bar{c}_{ri}\bar{c}_{sj}\bar{c}_{tk} \equiv C_{(4)}\beta^4 + C_{(3)}\beta^3 + C_{(2)}\beta^2 + C_{(1)}\beta + C_{(0)},$$

the coefficients in the quartic are

$$\begin{aligned} C_{(0)} &= -\bar{v}_i\bar{v}_j\bar{v}_k\bar{w}_r\bar{w}_s\bar{w}_t; \\ C_{(1)} &= a_{ir}a_{js}a_{kt} - \bar{a}_{ri}\bar{v}_j\bar{v}_k\bar{w}_s\bar{w}_t - \bar{a}_{sj}\bar{v}_i\bar{v}_k\bar{w}_r\bar{w}_t - \bar{a}_{tk}\bar{v}_i\bar{v}_j\bar{w}_r\bar{w}_s; \\ C_{(2)} &= a_{ir}a_{js}v_kw_t + a_{js}a_{kt}v_iw_r + a_{ir}a_{kt}v_jw_s - \bar{a}_{ri}\bar{a}_{sj}\bar{v}_k\bar{w}_t - \bar{a}_{sj}\bar{a}_{tk}\bar{v}_i\bar{w}_r - \bar{a}_{ri}\bar{a}_{tk}\bar{v}_j\bar{w}_s; \\ C_{(3)} &= a_{ir}v_jv_kv_sw_t + a_{js}v_iv_kw_rw_t + a_{kt}v_iv_jw_rw_s - \bar{a}_{ri}\bar{a}_{sj}\bar{a}_{tk}; \\ C_{(4)} &= v_iv_jv_kv_rw_sw_t, \end{aligned} \quad (6.9)$$

where

$$v_i = V_i/p^2, \quad w_r = W_r/p^2, \quad \bar{v}_i = \bar{V}_i/p^2, \quad \bar{w}_r = \bar{W}_r/p^2.$$

Now consideration of particular cases for the momenta demonstrates that the roots of  $\mathcal{C}(\beta)$  are irrational functions of the momenta and that rational results can only be obtained by summing over each of the roots. So, from (6.1), we take

$$\mathcal{M}^{\epsilon_1 \dots \epsilon_6} = \frac{1}{p^2} \oint_{\mathcal{O}} \frac{F^{\epsilon_1 \dots \epsilon_6}(c)}{\mathcal{C}(\beta)} d\beta, \quad (6.10)$$

where  $\mathcal{O}$  is taken to be a contour encircling each of these roots once, but none of the poles of  $F(c)$ . We now discuss the evaluation of the amplitude using contour manipulation, in the spirit of [1, 2].

Because the integrand tends to zero as  $\beta^{-3}$  or faster, as  $\beta \rightarrow \infty$ , we have

$$\mathcal{M}^{\epsilon_1 \dots \epsilon_6} = -\frac{1}{p^2} \oint_{\mathcal{O}'} \frac{F^{\epsilon_1 \dots \epsilon_6}(c)}{\mathcal{C}(\beta)} d\beta,$$

where the contour  $\mathcal{O}'$  encircles the poles in  $\beta$  of  $F^{\epsilon_1 \dots \epsilon_6}(c)$  positively, but excludes the zeros of  $\mathcal{C}(\beta)$ .

(a) For the case  $(i, t, j, r, k, s) \equiv (+, -, +, -, +, -)$ ,

$$\mathcal{M}^{+-+-+-} = \frac{1}{2\pi i} \oint_{\mathcal{O}_a} \frac{d\beta}{p^2 c_{ir} c_{js} c_{kt} \mathcal{C}(\beta)},$$

where  $\mathcal{O}_a$  encircles  $\beta = -A_{ir}/V_i W_r$ ,  $\beta = -A_{js}/V_j W_s$  and  $\beta = -A_{kt}/V_k W_t$ . The integral is the sum of the contributions of these poles of the integrand, and at each of them  $\mathcal{C}(\beta) = \beta c_{ir} c_{js} c_{kt} - \beta^3 \bar{c}_{ri} \bar{c}_{sj} \bar{c}_{tk} = -\beta^3 \bar{c}_{ri} \bar{c}_{sj} \bar{c}_{tk}$ , because  $c_{ir} c_{js} c_{kt} = 0$ . Thus, using formulae listed in appendix B,

$$\begin{aligned} \mathcal{M}^{+-+-+-} &= -\frac{1}{2\pi i} \oint_{\mathcal{O}_a} \frac{d\beta}{p^2 c_{ir} c_{js} c_{kt} \beta^3 \bar{c}_{ri} \bar{c}_{sj} \bar{c}_{tk}} \\ &= -\frac{[j, k]^4 \langle s, t \rangle^4}{\langle t | P_{si} | k \rangle \langle s | P_{ti} | j \rangle p_{jkr}^2 \langle i, s \rangle [r, j] \langle i, t \rangle [r, k]} + 2 \text{ cyclic terms}, \end{aligned} \quad (6.11)$$

where the two additional cyclic terms are obtained by simultaneously cyclically rotating  $(i, j, k)$  and  $(r, s, t)$ , and where we write

$$\langle \alpha_1 | P_{\beta_1 \dots \beta_l} | \alpha_2 \rangle = \langle \alpha_1, \beta_1 \rangle [\beta_1, \alpha_2] + \dots + \langle \alpha_1, \beta_l \rangle [\beta_l, \alpha_2], \quad (6.12)$$

so that

$$\langle t | P_{si} | k \rangle = -\langle t | P_{rj} | k \rangle, \text{ etc.},$$

and

$$p_{\alpha_1 \dots \alpha_l}^2 = (p_{\alpha_1} + \dots + p_{\alpha_l})^2. \quad (6.13)$$

Having written the amplitude in the form (6.11), we can replace the contour  $\mathcal{O}_a$  by a contour  $\mathcal{O}'_a$  encircling the poles corresponding to  $\bar{c}_{ri} = 0$ ,  $\bar{c}_{sj} = 0$  and  $\bar{c}_{tk} = 0$ , i.e.

$\beta = -\bar{V}_i \bar{W}_r / \bar{A}_{ir}$ ,  $\beta = -\bar{V}_j \bar{W}_s / \bar{A}_{js}$  and  $\beta = -\bar{V}_k \bar{W}_t / \bar{A}_{kt}$ , respectively:

$$\begin{aligned} \mathcal{M}^{+-+--+} &= \frac{1}{2\pi i} \oint_{\mathcal{O}'_a} \frac{d\beta}{p^2 c_{ir} c_{js} c_{kt} \beta^3 \bar{c}_{ri} \bar{c}_{sj} \bar{c}_{tk}} \\ &= -\frac{\bar{A}_{ri}^4}{\langle k | P_{si} | t \rangle \langle j | P_{ti} | s \rangle p_{jkr}^2 [i, s] \langle r, j \rangle [i, t] \langle r, k \rangle} + 2 \text{ cyclic terms,} \end{aligned} \quad (6.14)$$

where  $\bar{A}_{ri}$  is defined in (6.7). (6.14) provides a second, equivalent form for the alternating tree amplitude.

(b) For the case  $(k, i, s, t, j, r) \equiv (+, +, -, -, +, -)$ ,

$$\mathcal{M}^{++----} = -\frac{1}{2\pi i} \oint_{\mathcal{O}_b} \frac{c_{is} d\beta}{p^2 c_{ir} c_{js} \beta \bar{c}_{rj} \mathcal{C}(\beta)},$$

where  $\mathcal{O}_b$  encircles  $\beta = -A_{ir}/V_i W_r$ ,  $\beta = -A_{js}/V_j W_s$  and  $\beta = -\bar{V}_j \bar{W}_r / \bar{A}_{rj}$ . The integral is the sum of the contributions of these poles of the integrand, and at each of them  $\mathcal{C}(\beta) = \beta^2 [c_{is} c_{jr} \bar{c}_{ri} \bar{c}_{sj} - c_{ir} c_{js} \bar{c}_{si} \bar{c}_{rj}] = \beta^2 c_{is} c_{jr} \bar{c}_{ri} \bar{c}_{sj}$ , because  $c_{ir} c_{js} \bar{c}_{si} \bar{c}_{rj} = 0$  at the poles. Thus

$$\begin{aligned} \mathcal{M}^{++----} &= -\frac{1}{2\pi i} \oint_{\mathcal{O}_b} \frac{d\beta}{p^2 c_{ir} c_{js} \beta \bar{c}_{rj} c_{jr} \beta^2 \bar{c}_{ri} \bar{c}_{sj}} \\ &= -\frac{[j, k]^4 \langle s, t \rangle^3}{\langle t | P_{si} | k \rangle \langle i | P_{rk} | j \rangle [k, r] p_{jkr}^2 \langle i, s \rangle [r, j]} - \frac{[k, i]^3 \langle t, r \rangle^4}{\langle t | P_{rj} | k \rangle \langle r | P_{tj} | s \rangle [i, s] p_{kis}^2 \langle j, t \rangle \langle j, r \rangle} \\ &\quad - \frac{\bar{A}_{rj}^4}{\langle i | P_{rk} | j \rangle \langle r | P_{tj} | s \rangle p_{kir}^2 \langle k, r \rangle [j, t] [s, t] \langle k, i \rangle}. \end{aligned} \quad (6.15)$$

For an alternative form, starting from (6.15), we can replace the contour  $\mathcal{O}_b$  by a contour  $\mathcal{O}'_b$  encircling the poles corresponding to  $\bar{c}_{ri} = 0$ ,  $\bar{c}_{sj} = 0$  and  $c_{jr} = 0$ , i.e.  $\beta = -\bar{V}_i \bar{W}_r / \bar{A}_{ir}$ ,  $\beta = -\bar{V}_j \bar{W}_s / \bar{A}_{js}$  and  $\beta = -A_{jr}/V_j W_r$ , respectively:

$$\begin{aligned} \mathcal{M}^{++----} &= \frac{1}{2\pi i} \oint_{\mathcal{O}'_b} \frac{d\beta}{p^2 c_{ir} c_{js} \beta \bar{c}_{rj} c_{jr} \beta^2 \bar{c}_{ri} \bar{c}_{sj}} \\ &= -\frac{\bar{A}_{ri}^4}{\langle k | P_{si} | t \rangle \langle j | P_{rk} | i \rangle \langle k, r \rangle p_{jkr}^2 [i, s] \langle r, j \rangle [s, t]} \\ &\quad - \frac{\bar{A}_{sj}^4}{\langle k | P_{rj} | t \rangle \langle s | P_{tj} | r \rangle \langle i, s \rangle p_{kis}^2 [j, t] [j, r] \langle k, i \rangle} \\ &\quad - \frac{[k, i]^4 \langle s, t \rangle^4}{\langle j | P_{rk} | i \rangle \langle s | P_{tj} | r \rangle p_{kir}^2 [k, r] \langle j, t \rangle \langle s, t \rangle [k, i]}. \end{aligned} \quad (6.17)$$

(c) For the case  $(i, j, k, r, s, t) \equiv (+, +, +, -, -, -)$ ,

$$\mathcal{M}^{+++---} = \frac{1}{2\pi i} \oint_{\mathcal{O}_c} \frac{c_{js} d\beta}{p^2 \beta^2 \bar{c}_{tk} \bar{c}_{ri} \mathcal{C}(\beta)},$$

where  $\mathcal{O}_c$  encircles  $\beta = -\bar{V}_k \bar{W}_t / \bar{A}_{tk}$  and  $\beta = -\bar{V}_i \bar{W}_r / \bar{A}_{ri}$ . The integral is the sum of the contributions of these poles of the integrand, and at each of them  $\mathcal{C}(\beta) =$

$$\begin{aligned}
 \beta c_{ir} c_{js} c_{kt} - \beta^3 \bar{c}_{ri} \bar{c}_{sj} \bar{c}_{tk} &= \beta c_{ir} c_{js} c_{kt}, \text{ because } \bar{c}_{ri} \bar{c}_{sj} \bar{c}_{tk} = 0 \text{ at the poles. Thus} \\
 \mathcal{M}^{+++---} &= \frac{1}{2\pi i} \oint_{\mathcal{O}_c} \frac{d\beta}{p^2 \beta^2 \bar{c}_{tk} \bar{c}_{ri} \beta c_{ir} c_{kt}} \\
 &= -\frac{\bar{A}_{ri}^3}{\langle j, k \rangle [s, t] \langle j | P_{ti} | s \rangle p_{jkr}^2 [i, t] \langle r, k \rangle} - \frac{\bar{A}_{tk}^3}{\langle j, i \rangle [s, r] \langle j | P_{rk} | s \rangle p_{jit}^2 [k, r] \langle t, i \rangle}.
 \end{aligned} \tag{6.18}$$

To find the alternative form, starting from (6.18), we can replace the contour  $\mathcal{O}_c$  by a contour  $\mathcal{O}'_c$  encircling the poles corresponding to  $c_{ir} = 0$ ,  $c_{kt} = 0$  and  $\beta = 0$ , i.e.  $\beta = -A_{ir}/V_i W_r$ ,  $\beta = -A_{kt}/V_k W_t$  as well as  $\beta = 0$ , respectively:

$$\begin{aligned}
 \mathcal{M}^{+++---} &= -\frac{1}{2\pi i} \oint_{\mathcal{O}'_c} \frac{d\beta}{p^2 \beta^2 \bar{c}_{tk} \bar{c}_{ri} \beta c_{ir} c_{kt}} \\
 &= \frac{[j, k]^3 \langle s, t \rangle^3}{A_{ir} \langle s | P_{ti} | j \rangle p_{jkr}^2 \langle i, t \rangle [r, k]} + \frac{[i, j]^3 \langle r, s \rangle^3}{A_{kt} \langle s | P_{kr} | j \rangle p_{ijt}^2 \langle r, k \rangle [i, t]} \\
 &\quad - \frac{(p^2)^3}{\langle ij \rangle [r, s] \langle j, k \rangle [st] A_{ir} A_{kt}},
 \end{aligned} \tag{6.19}$$

where the  $A_{ir}$  is defined in (5.5).

## 7 Multiple constraints and an NMHV 7-point function

We now consider the general situation in which the number of constraints,  $N_R > 1$ . The general form of the amplitude is

$$\mathcal{M}^{\epsilon_1 \dots \epsilon_N} = \tilde{K} \oint F(c) \prod_{\substack{k \in \mathcal{P}' \\ t \in \mathcal{N}'}} \frac{d\beta_{kt}}{\mathcal{C}_{kt}(\beta)}, \tag{7.1}$$

where  $\tilde{K}$  is given by (5.19). We must specify further the multi-dimensional contour in (7.1) and explain how it may be evaluated. We remember, from the end of section 3, that the motivation for (7.1), is obtained by considering the result (3.11) of evaluating the delta function constraints  $\delta^{N_R}(\hat{\mathcal{C}})$  in (3.9). By replacing real integrals over delta functions with contour integrals around corresponding poles, we effectively replaced an expression involving the inverse of the modulus of the Jacobian of the constraints  $\hat{\mathcal{C}}$ , summed over a discrete set of points corresponding to the simultaneous solution of the constraint conditions, with the same expression with the modulus removed, so that the sum over the simultaneous solutions of the constraints now includes the phases of the inverse Jacobian at these points. We saw in section 6, in the case of the  $N_R = 1$  6-point function, that this interpretation was forced on us if we were to be able to reproduce the rational form for the amplitude known from gauge theory.

In order to specify (7.1) more precisely, we choose an order for the constraints,  $\mathcal{C}_K$ ,  $K = 1 \dots N_R$ , and order the parameters,  $\beta_K$ , correspondingly. Then, as in section 5.1 of [2], the contribution of a particular simultaneous solution  $\beta = \beta^0$  of the constraint equations,  $\mathcal{C}_K(\beta) = 0$ , is

$$\tilde{K} \oint_{\mathcal{O}_{\beta^0}} F(c) \left[ \prod_{K=1}^{N_R} \frac{1}{\mathcal{C}_K(\beta)} \right] d\beta = \tilde{K} F(c) \left[ \frac{\partial(\mathcal{C})}{\partial(\beta)} \right]^{-1} \Big|_{\beta=\beta^0}, \tag{7.2}$$

where  $d\beta = d\beta_1 \wedge \dots \wedge d\beta_{N_R}$  and the contour  $\mathcal{O}^0$  is chosen to be a surface of the form  $\{\beta : |\mathcal{C}_K(\beta)| = \epsilon, 1 \leq K \leq N_R\}$ , with its orientation determined by the order of the  $\mathcal{C}_K$  (as in [2]), enclosing  $\beta = \beta^0$  but no other zero of the  $\mathcal{C}_K$ . The integral (7.2) is the residue of the integrand at  $\beta = \beta_0$  and it is antisymmetric under independent permutations of the order of the  $\mathcal{C}_K$  or of the  $\beta_K$ , and so symmetric under simultaneous identical permutations of both.

Now, as in the  $N_R = 1$  case, we should sum (7.2) over all the simultaneous solutions,  $\beta = \beta^0$ , of the constraints,  $\mathcal{C}_K(\beta) = 0$ , but exclude the contribution of other poles of the integrand arising from  $F(c)$ . At this point we should note that, with the particular set of constraints we have chosen in section 4, namely  $\{\mathcal{C}_{kt} : k \neq I, J, t \neq R, S\}$ , there are always four ‘trivial’ or ‘spurious’ simultaneous solutions of the constraints, namely  $c_{IR} = c_{JR} = 0$ ;  $c_{IS} = c_{JS} = 0$ ;  $c_{IR} = c_{IS} = 0$ ; and  $c_{JR} = c_{JS} = 0$ . They are introduced when we move from statements about the matrix (1.6), with entries  $c_{ir}^{-1}$  to statements about multinomials (1.7) in the link variables,  $c_{ir}$ , by multiplying by products of them. They do not correspond to the matrix (1.6) having rank 2; in fact, they are artifacts of the particular choice of the independent set of constraints. These spurious solutions should be excluded from the sum.

The problem again with trying to evaluate this sum directly is that the solutions to the constraints are not all rational individually, only their sum is. So, we seek to use a multidimensional version of the contour manipulation arguments we used in section 6 to evaluate the amplitude to obtain the familiar rational results. As in [2], this is provided by the global residue theorem. Consider an  $N$ -dimensional integral of the form

$$\oint \frac{\Phi(\beta)d\beta}{\prod_{\alpha=1}^M h_{\alpha}(\beta)}, \tag{7.3}$$

where  $N < M$ . For distinct indices  $\alpha_1, \dots, \alpha_N$ , the residue of the integrand at a common zero  $\beta^0$  of  $h_{\alpha_1}, \dots, h_{\alpha_N}$ , assumed to be a simple zero, is

$$\mathcal{R}(h_{\alpha_1}, \dots, h_{\alpha_N}) = \Phi(\beta^0) \prod_{\alpha \notin A} \frac{1}{h_{\alpha}(\beta^0)} \left[ \frac{\partial(h_{\alpha_1}, \dots, h_{\alpha_N})}{\partial(\beta_1, \dots, \beta_N)} \right]_{\beta=\beta^0}^{-1}, \tag{7.4}$$

where  $A = \{\alpha_1, \dots, \alpha_N\}$ . It may be that the functions  $h_{\alpha_1}, \dots, h_{\alpha_N}$  have more than one common zero but we shall assume the set of such simultaneous zeros is finite and, if there is more than one, we shall understand  $\mathcal{R}(h_{\alpha_1}, \dots, h_{\alpha_N})$  to denote the sum of the residues (7.4) at these simultaneous zeros. Suppose  $\Gamma_{\ell}, 1 \leq \ell \leq N$  are disjoint subsets of  $\{1, \dots, M\}$ , whose union is the whole set. Then a version [2] of the global residue theorem states that

$$\sum_{\alpha_{\ell} \in \Gamma_{\ell}} \mathcal{R}(h_{\alpha_1}, \dots, h_{\alpha_N}) = 0. \tag{7.5}$$

We demonstrate the effectiveness of this for evaluating amplitudes by considering a 7-point NMHV tree with helicities  $(i, r, k, s, l, t, j) \equiv (+ - + - + - +)$ , so that  $m = 4$ ,  $n = 3$ ,  $N_R = 2$ . We take the choice  $(I, J, R, S) = (i, j, r, s)$ . In this case there are two constraints,  $\mathcal{C}_{kt}, \mathcal{C}_{lt}$ , and, as in (5.16), we can take the two integration variables to be

$\beta_k = \beta^{ijk}$ ,  $\beta_l = \beta^{ijl}$ , where, for  $r \in \mathcal{N}$ ,

$$c_{ir} = a_{ir} + (\beta_k[j, k] + \beta_l[j, l])W_r/p^2, \quad (7.6)$$

$$c_{jr} = a_{jr} + (\beta_k[k, i] + \beta_l[l, i])W_r/p^2, \quad (7.7)$$

$$c_{kr} = a_{kr} + \beta_k[i, j]W_r/p^2, \quad c_{lr} = a_{lr} + \beta_l[i, j]W_r/p^2, \quad r \in \mathcal{N}. \quad (7.8)$$

From (4.9),

$$F(c) = \frac{c_{ir}c_{jt}}{c_{kt}c_{lr}}, \quad (7.9)$$

and so, from (5.18), the contour expression for the amplitude is of the form

$$\mathcal{M} = \frac{[i, j]}{(p^2)^2} \oint_{\mathcal{O}} \frac{c_{ir}c_{jt}}{c_{kt}c_{lr}c_{kt}c_{lt}} d\beta_k d\beta_l. \quad (7.10)$$

In (7.10) the contour  $\mathcal{O}$  is chosen so that it includes the residue contributions from each of the simultaneous solutions of the constraints  $\mathcal{C}_{kt} = \mathcal{C}_{lt} = 0$ , but excluding the ‘spurious’ solutions.  $\mathcal{C}_{kt} = \mathcal{C}_{lt} = 0$  if  $c_{ir} = c_{jr} = 0$  or  $c_{is} = c_{js} = 0$  or  $c_{it} = c_{jt} = 0$ ; we do not have to consider possible spurious contributions from, e.g.  $c_{ir} = c_{is} = 0$ , because the variables  $c_{ir}, c_{is}, c_{it}$  are not independent and no two of them can vanish together for general momenta, and similar considerations apply to  $c_{jr}, c_{js}, c_{jt}$ . The residues at  $c_{ir} = c_{jr} = 0$  and  $c_{it} = c_{jt} = 0$  vanish because of the presence of  $c_{ir}$  and  $c_{jt}$ , respectively, in the numerator of the integrand, so that we only have to exclude the contribution from  $c_{is} = c_{js} = 0$  explicitly.

Applying the global residue theorem taking  $\Gamma_1$  to correspond to  $\{\mathcal{C}_{kt}, c_{lr}\}$  and  $\Gamma_2$  to correspond to  $\{\mathcal{C}_{lt}, c_{kt}\}$ , we obtain

$$\mathcal{R}(\mathcal{C}_{kt}, \mathcal{C}_{lt}) + \mathcal{R}(c_{lr}, \mathcal{C}_{lt}) + \mathcal{R}(\mathcal{C}_{kt}, c_{kt}) + \mathcal{R}(c_{lr}, c_{kt}) = 0. \quad (7.11)$$

and so, excluding the ‘spurious’ contribution,

$$\begin{aligned} \mathcal{M} &= \mathcal{R}(\mathcal{C}_{kt}, \mathcal{C}_{lt}) - \mathcal{R}(c_{is}, c_{js}) \\ &= -\mathcal{R}(c_{lr}, c_{kt}) - \mathcal{R}(\mathcal{C}_{kt}, c_{kt}) - \mathcal{R}(c_{lr}, \mathcal{C}_{lt}) - \mathcal{R}(c_{is}, c_{js}) \end{aligned} \quad (7.12)$$

The choice of  $\Gamma_1$  and  $\Gamma_2$  has been made so that  $\mathcal{R}(\mathcal{C}_{kt}, c_{kt})$  and  $\mathcal{R}(c_{lr}, \mathcal{C}_{lt})$  are as easy to evaluate as  $\mathcal{R}(c_{lr}, c_{kt})$  and  $\mathcal{R}(c_{is}, c_{js})$  are.

Because

$$\begin{aligned} \mathcal{C}_{kt} &= c_{ir}c_{js}c_{kt}c_{jr}c_{ks}c_{it} + c_{jr}c_{ks}c_{it}c_{kr}c_{is}c_{jt} + c_{kr}c_{is}c_{jt}c_{ir}c_{js}c_{kt} \\ &\quad - c_{kr}c_{js}c_{it}c_{jr}c_{is}c_{kt} - c_{jr}c_{is}c_{kt}c_{ir}c_{ks}c_{jt} - c_{ir}c_{ks}c_{jt}c_{kr}c_{js}c_{it} \\ &= c_{is}c_{kt}c_{rs}^{jk}c_{tr}^{ij} - c_{it}c_{ks}c_{tr}^{jk}c_{rs}^{ij}, \end{aligned} \quad (7.13)$$

where

$$c_{rs}^{ij} = c_{ir}c_{js} - c_{is}c_{jr}, \quad (7.14)$$

we have that

$$\mathcal{C}_{kt}|_{c_{kt}=0} = c_{kr}c_{ks}c_{it}c_{jt}(c_{is}c_{jr} - c_{ir}c_{js}), \quad (7.15)$$

from which it follows that we can write  $\mathcal{R}(\mathcal{C}_{kt}, c_{kt})$  as a sum of terms corresponding to the factors of  $\mathcal{C}_{kt}$  when  $c_{kt} = 0$ . Because, in this case, for general momenta,  $c_{kr}, c_{ks} \neq 0$  when  $c_{kt} = 0$ , and because of the presence of  $c_{jt}$  in the numerator, we only have to consider the factors  $c_{it}$  and  $c_{rs}^{ij}$ , implying

$$\mathcal{R}(\mathcal{C}_{kt}, c_{kt}) = \mathcal{R}(c_{it}, c_{kt}) + \mathcal{R}(c_{rs}^{ij}, c_{kt}). \quad (7.16)$$

Similarly, from

$$\mathcal{C}_{lt}|_{c_{lr}=0} = c_{ls}c_{lt}c_{ir}c_{jr}(c_{it}c_{js} - c_{is}c_{jt}), \quad (7.17)$$

we deduce

$$\mathcal{R}(c_{lr}, \mathcal{C}_{lt}) = \mathcal{R}(c_{lr}, c_{jr}) + \mathcal{R}(c_{lr}, c_{st}^{ij}). \quad (7.18)$$

Combining (7.16) and (7.18) with (7.12), we have  $\mathcal{M}$  expressed as a sum of six terms,

$$\mathcal{M} = -\mathcal{R}(c_{lr}, c_{kt}) - \mathcal{R}(c_{it}, c_{kt}) - \mathcal{R}(c_{rs}^{ij}, c_{kt}) - \mathcal{R}(c_{lr}, c_{jr}) - \mathcal{R}(c_{lr}, c_{st}^{ij}) - \mathcal{R}(c_{is}, c_{js}). \quad (7.19)$$

We now calculate each of these terms in turn to obtain  $\mathcal{M}$ .

Using formulae from appendix C, we find the following:

(a) residue from  $c_{lr} = c_{kt} = 0$

$$\mathcal{R}(c_{lr}, c_{kt}) = \frac{[i, j]}{(p^2)^2} \left[ \frac{\partial(c_{lr}, c_{kt})}{\partial(\beta_k, \beta_l)} \right]^{-1} \frac{c_{ir}c_{jt}}{\mathcal{C}_{kt}\mathcal{C}_{lt}} \quad (7.20)$$

$$= -\frac{1}{[i, j]W_tW_r c_{kr}c_{ks}c_{it}c_{ls}c_{lt}c_{jr}(c_{is}c_{jr} - c_{ir}c_{js})(c_{it}c_{js} - c_{is}c_{jt})} \\ = -\frac{W_t^4 W_r^4 [i, j]^3}{\langle k, r \rangle \langle k, s \rangle \langle l, s \rangle \langle l, t \rangle \langle s | P_{lt} | j \rangle \langle s | P_{kr} | i \rangle \langle t | P_{ij} P_{kr} | s \rangle \langle s | P_{ij} P_{lt} | r \rangle}. \quad (7.21)$$

(b) residue from  $c_{it} = c_{kt} = 0$  Since at  $c_{it} = c_{kt} = 0$ ,

$$\frac{\partial(\mathcal{C}_{kt}, c_{kt})}{\partial(\beta_k, \beta_l)} = -\frac{\partial\mathcal{C}_{kt}}{\partial\beta_l} \frac{\partial c_{kt}}{\partial\beta_k} = -\frac{\partial c_{kt}}{\partial\beta_k} \frac{\partial c_{it}}{\partial\beta_l} c_{kr}c_{ks}c_{jt}(c_{is}c_{jr} - c_{ir}c_{js}),$$

$$\mathcal{R}(c_{it}, c_{kt}) = \frac{[i, j]}{(p^2)^2} \left[ \frac{\partial(\mathcal{C}_{kt}, c_{kt})}{\partial(\beta_k, \beta_l)} \right]^{-1} \frac{c_{ir}c_{jt}}{c_{lr}\mathcal{C}_{lt}} \quad (7.22)$$

$$= -\frac{1}{[j, l]W_t^2 c_{lr}c_{kr}c_{ks}c_{is}c_{jt}c_{lt}(c_{is}c_{jr} - c_{ir}c_{js})(c_{js}c_{lr} - c_{jr}c_{ls})} \\ = \frac{W_t^4 [j, l]^4}{p_{ltj}^2 \langle i, r \rangle \langle r, k \rangle \langle k, s \rangle [l, t] [t, j] \langle s | P_{lt} | j \rangle \langle i | P_{jt} | l \rangle}. \quad (7.23)$$



(c) residue from  $c_{rs}^{ij} = c_{kt} = 0$  Since at  $c_{rs}^{ij} = c_{kt} = 0$ ,

$$\begin{aligned} \frac{\partial(\mathcal{C}_{kt}, c_{kt})}{\partial(\beta_k, \beta_l)} &= -\frac{\partial\mathcal{C}_{kt}}{\partial\beta_l} \frac{\partial c_{kt}}{\partial\beta_k} = \frac{\partial c_{kt}}{\partial\beta_k} \frac{\partial c_{rs}^{ij}}{\partial\beta_l} c_{kr} c_{ks} c_{it} c_{jt}, \\ \frac{\partial c_{rs}^{ij}}{\partial\beta_l} &= [(c_{js} W_r - c_{jr} W_s)[j, l] + (c_{ir} W_s - c_{is} W_r)[l, i]]/p^2 = -\langle t|P_{ij}|l\rangle/p^2 \\ \mathcal{C}_{lt}|_{c_{rs}^{ij}=0} &= c_{is} c_{lt} c_{tr}^{ij} c_{rs}^{jl}. \end{aligned} \quad (7.24)$$

$$\mathcal{R}(c_{rs}^{ij}, c_{kt}) = \frac{[i, j]}{(p^2)^2} \left[ \frac{\partial(\mathcal{C}_{kt}, c_{kt})}{\partial(\beta_k, \beta_l)} \right]^{-1} \frac{c_{ir} c_{jt}}{c_{lr} c_{lt}} \quad (7.25)$$

$$\begin{aligned} &= -\frac{c_{ir}}{\langle t|P_{ij}|l\rangle W_t c_{lr} c_{kr} c_{ks} c_{it} c_{is} c_{lt} (c_{jr} c_{it} - c_{jt} c_{ir}) (c_{jr} c_{ls} - c_{js} c_{lr})} \\ &= \frac{W_t^4 \langle t|P_{ij}|l\rangle^4}{p_{ijt}^2 p_{krs}^2 \langle r, k\rangle \langle k, s\rangle \langle i, j\rangle \langle j, t\rangle \langle r|P_{ks}|l\rangle \langle i|P_{jt}|l\rangle \langle t|P_{ij} P_{kr}|s\rangle}. \end{aligned} \quad (7.26)$$

(d) residue from  $c_{lr} = c_{jr} = 0$  Since at  $c_{lr} = c_{jr} = 0$ ,

$$\begin{aligned} \frac{\partial(c_{lr}, \mathcal{C}_{lt})}{\partial(\beta_k, \beta_l)} &= -\frac{\partial c_{lr}}{\partial\beta_l} \frac{\partial\mathcal{C}_{lt}}{\partial\beta_k} = \frac{\partial c_{jr}}{\partial\beta_k} \frac{\partial c_{lr}}{\partial\beta_l} c_{ir} c_{ls} c_{lt} (c_{js} c_{it} - c_{is} c_{jt}), \\ \mathcal{R}(c_{lr}, c_{jr}) &= \frac{[i, j]}{(p^2)^2} \left[ \frac{\partial(c_{lr}, \mathcal{C}_{lt})}{\partial(\beta_k, \beta_l)} \right]^{-1} \frac{c_{ir} c_{jt}}{c_{kt} c_{kt}} \end{aligned} \quad (7.27)$$

$$\begin{aligned} &= -\frac{1}{[k, i] W_r^2 c_{ls} c_{lt} c_{ir} c_{kr} c_{js} c_{kt} (c_{js} c_{it} - c_{is} c_{jt}) (c_{is} c_{kt} - c_{it} c_{ks})} \\ &= \frac{W_r^4 [k, i]^4}{p_{irk}^2 \langle s, l\rangle \langle l, t\rangle \langle t, j\rangle [i, r] [r, k] \langle j|P_{ir}|k\rangle \langle s|P_{kr}|i\rangle}. \end{aligned} \quad (7.28)$$

(e) residue from  $c_{lr} = c_{st}^{ij} = 0$  Since at  $c_{lr} = c_{st}^{ij} = 0$ ,

$$\begin{aligned} \frac{\partial(c_{lr}, \mathcal{C}_{lt})}{\partial(\beta_k, \beta_l)} &= -\frac{\partial c_{lr}}{\partial\beta_l} \frac{\partial\mathcal{C}_{lt}}{\partial\beta_k} = \frac{\partial c_{st}^{ij}}{\partial\beta_k} \frac{\partial c_{lr}}{\partial\beta_l} c_{ls} c_{lt} c_{ir} c_{jr}, \\ \frac{\partial c_{st}^{ij}}{\partial\beta_k} &= [(c_{is} W_t - c_{it} W_s)[k, i] + (c_{jt} W_s - c_{js} W_t)[j, k]]/p^2 = -\langle r|P_{ij}|k\rangle/p^2 \\ \mathcal{C}_{kt}|_{c_{st}^{ij}=0} &= c_{it} c_{kr} c_{st}^{jk} c_{rs}^{ij}. \end{aligned} \quad (7.29)$$

$$\mathcal{R}(c_{lr}, c_{st}^{ij}) = \frac{[i, j]}{(p^2)^2} \left[ \frac{\partial(c_{lr}, \mathcal{C}_{lt})}{\partial(\beta_k, \beta_l)} \right]^{-1} \frac{c_{ir} c_{jt}}{c_{kt} c_{kt}} \quad (7.30)$$

$$\begin{aligned} &= -\frac{c_{jt}}{\langle r|P_{ij}|k\rangle W_r c_{jr} c_{ls} c_{lt} c_{kt} c_{it} c_{kr} (c_{ir} c_{js} - c_{is} c_{jr}) (c_{js} c_{kt} - c_{ks} c_{jt})} \\ &= \frac{W_r^4 \langle r|P_{ij}|k\rangle^4}{p_{ijr}^2 p_{lst}^2 \langle ji\rangle \langle ir\rangle \langle sl\rangle \langle lt\rangle \langle j|P_{ir}|k\rangle \langle t|P_{ls}|k\rangle \langle s|P_{lt} P_{ij}|r\rangle}. \end{aligned} \quad (7.31)$$

(f) residue from  $c_{is} = c_{js} = 0$  Since at  $c_{is} = c_{js} = 0$ ,

$$\begin{aligned} \frac{\partial(\mathcal{C}_{kt}, \mathcal{C}_{lt})}{\partial(\beta_k, \beta_l)} &= \frac{\partial(\mathcal{C}_{kt}, \mathcal{C}_{lt})}{\partial(c_{is}, c_{js})} \cdot \frac{\partial(c_{is}, c_{js})}{\partial(\beta_k, \beta_l)}, \\ \frac{\partial(\mathcal{C}_{kt}, \mathcal{C}_{lt})}{\partial(c_{is}, c_{js})} &= c_{ir}c_{jr}c_{ks}c_{ls}c_{it}c_{jt}(c_{it}c_{jr} - c_{jt}c_{ir})(c_{kr}c_{lt} - c_{kt}c_{lr}) \\ \frac{\partial(c_{is}, c_{js})}{\partial(\beta_k, \beta_l)} &= \frac{[i, j][k, l]W_s^2}{(p^2)^2}, \\ \mathcal{R}(c_{is}, c_{js}) &= \frac{[i, j]}{(p^2)^2} \left[ \frac{\partial(\mathcal{C}_{kt}, \mathcal{C}_{lt})}{\partial(\beta_k, \beta_l)} \right]^{-1} \frac{c_{ir}c_{jt}}{c_{lr}c_{kt}} \end{aligned} \quad (7.32)$$

$$\begin{aligned} &= \frac{1}{[k, l]W_s^2 c_{jr}c_{lr}c_{ks}c_{ls}c_{it}c_{kt}(c_{it}c_{jr} - c_{jt}c_{ir})(c_{kr}c_{lt} - c_{kt}c_{lr})} \\ &= \frac{1}{p_{kls}^2 \langle t, j \rangle \langle j, i \rangle \langle i, r \rangle [l, s][s, k] \langle r | P_{ks} | l \rangle \langle t | P_{ls} | k \rangle}. \end{aligned} \quad (7.33)$$

Combining these results for the residues, we have that

$$\begin{aligned} \mathcal{M} &= -\mathcal{R}(c_{lr}, c_{kt}) - \mathcal{R}(c_{it}, c_{kt}) - \mathcal{R}(c_{rs}^{ij}, c_{kt}) - \mathcal{R}(c_{lr}, c_{jr}) - \mathcal{R}(c_{lr}, c_{st}^{ij}) - \mathcal{R}(c_{is}, c_{js}) \\ &= \frac{W_t^4 W_r^4 [i, j]^3}{\langle k, r \rangle \langle k, s \rangle \langle l, s \rangle \langle l, t \rangle \langle s | P_{lt} | j \rangle \langle s | P_{kr} | i \rangle \langle t | P_{ij} P_{kr} | s \rangle \langle s | P_{ij} P_{lt} | r \rangle} \\ &\quad - \frac{W_t^4 [j, l]^4}{p_{ltj}^2 \langle i, r \rangle \langle r, k \rangle \langle k, s \rangle [l, t][t, j] \langle s | P_{lt} | j \rangle \langle i | P_{jt} | l \rangle} \\ &\quad - \frac{W_t^4 \langle t | P_{ij} | l \rangle^4}{p_{ijr}^2 p_{krs}^2 \langle r, k \rangle \langle k, s \rangle \langle i, j \rangle \langle j, t \rangle \langle r | P_{ks} | l \rangle \langle i | P_{jt} | l \rangle \langle t | P_{ij} P_{kr} | s \rangle} \\ &\quad - \frac{W_r^4 [k, i]^4}{p_{irk}^2 \langle s, l \rangle \langle l, t \rangle \langle t, j \rangle [i, r][r, k] \langle j | P_{ir} | k \rangle \langle s | P_{kr} | i \rangle} \\ &\quad - \frac{W_r^4 \langle r | P_{ij} | k \rangle^4}{p_{ijr}^2 p_{lst}^2 \langle j | i \rangle \langle i | r \rangle \langle s | l \rangle \langle l | t \rangle \langle j | P_{ir} | k \rangle \langle t | P_{ls} | k \rangle \langle s | P_{lt} P_{ij} | r \rangle} \\ &\quad - \frac{W_s^4 [k, l]^4}{p_{kls}^2 \langle t, j \rangle \langle j, i \rangle \langle i, r \rangle [l, s][s, k] \langle r | P_{ks} | l \rangle \langle t | P_{ls} | k \rangle}. \end{aligned} \quad (7.34)$$

The expression (7.34) has the appropriate soft limits and is antisymmetric under the transformations  $i \leftrightarrow j$ ,  $k \leftrightarrow l$ ,  $r \leftrightarrow t$ ,  $s \leftrightarrow s$ . It is equal, up to notation and signs, to the expressions computed directly from the field theory, and by recursion relations [19, 20].

Other tree amplitudes can be computed in a similar fashion from the expressions for the integrand in section 4. It seems that the dual S-matrix of ACCK leads back to twistor string theory at tree level. It will be interesting to pursue this link at the loop level.

## Acknowledgments

We are grateful to Nima Arkani-Hamed for discussions, who also informed us that some similar results had been obtained by Marcus Spradlin and Anastasia Volovich [27]. LD thanks the Institute for Advanced Study at Princeton for its hospitality. LD was partially supported by the U.S. Department of Energy, Grant No. DE-FG02-06ER-4141801, Task A.

## A Interchange of particles between $\mathcal{P}$ and $\mathcal{N}$

For  $k \in \mathcal{P}$ ,  $t \in \mathcal{N}$ , let  $\mathcal{P}_0 = \mathcal{P} \sim \{k\}$ ,  $\mathcal{N}_0 = \mathcal{N} \sim \{t\}$ , and  $\mathcal{P}' = \mathcal{P}_0 \cup \{t\}$ ,  $\mathcal{N}' = \mathcal{N}_0 \cup \{k\}$ ,

$$\begin{aligned}\pi_i &= \sum_{r \in \mathcal{N}_0} c_{ir} \pi_r + c_{it} \pi_t, & i \in \mathcal{P} \\ c_{kt} \pi_i - c_{it} \pi_k &= \sum_{r \in \mathcal{N}_0} (c_{ir} c_{kt} - c_{it} c_{kr}) \pi_r, & i \in \mathcal{P}_0 \\ \pi_i &= \sum_{r \in \mathcal{N}_0} \frac{(c_{ir} c_{kt} - c_{it} c_{kr})}{c_{kt}} \pi_r + \frac{c_{it}}{c_{kt}} \pi_k, & i \in \mathcal{P}_0 \\ \pi_t &= - \sum_{r \in \mathcal{N}_0} \frac{c_{kr}}{c_{kt}} \pi_r + \frac{1}{c_{kt}} \pi_k.\end{aligned}$$

So

$$\pi_i = \sum_{r \in \mathcal{N}'} \tilde{c}_{ir} \pi_r, \quad r \in \mathcal{P}',$$

with

$$\tilde{c}_{ir} = \frac{c_{ir} c_{kt} - c_{it} c_{kr}}{c_{kt}}, \quad i \in \mathcal{P}_0, r \in \mathcal{N}_0; \quad \tilde{c}_{ik} = \frac{c_{it}}{c_{kt}}, \quad i \in \mathcal{P}_0; \quad \tilde{c}_{tr} = -\frac{c_{kr}}{c_{kt}}, \quad r \in \mathcal{N}_0; \quad \tilde{c}_{tk} = \frac{1}{c_{kt}}.$$

Similarly

$$-\bar{\pi}_r = \sum_{i \in \mathcal{P}'} \bar{\pi}_i \tilde{c}_{ir}, \quad r \in \mathcal{N}',$$

with the same definition of  $\tilde{c}_{ir}$ . It remains to show that the  $\tilde{c}_{ir}$  satisfy the same constraints as the  $c_{ir}$ . To do this we establish formulae for them in terms of the  $\kappa_\ell, \rho_\ell$ .

$$\begin{aligned}\tilde{c}_{js} &= \frac{c_{js} c_{kt} - c_{jt} c_{ks}}{c_{kt}} = \frac{\kappa_j}{\kappa_s} \left[ \prod_{\substack{r \in \mathcal{N} \\ r \neq s}} \frac{\rho_j - \rho_r}{\rho_s - \rho_r} - \prod_{\substack{r \in \mathcal{N} \\ r \neq t}} \frac{\rho_j - \rho_r}{\rho_k - \rho_r} \prod_{\substack{r \in \mathcal{N} \\ u \neq s}} \frac{\rho_k - \rho_r}{\rho_s - \rho_u} \right], & j \in \mathcal{P}_0, s \in \mathcal{N}_0, \\ &= \frac{\kappa_j}{\kappa_s} \left[ \frac{\rho_j - \rho_t}{\rho_s - \rho_t} \cdot \frac{\rho_s - \rho_k}{\rho_j - \rho_k} - \frac{\rho_j - \rho_s}{\rho_k - \rho_s} \cdot \frac{\rho_k - \rho_t}{\rho_s - \rho_t} \cdot \frac{\rho_s - \rho_k}{\rho_j - \rho_k} \right] \prod_{\substack{r \in \mathcal{N}' \\ r \neq s}} \frac{\rho_j - \rho_r}{\rho_s - \rho_r} \\ &= \frac{\kappa_j}{\kappa_s} \prod_{\substack{r \in \mathcal{N}' \\ r \neq s}} \frac{\rho_j - \rho_r}{\rho_s - \rho_r}, & j \in \mathcal{P}_0, s \in \mathcal{N}_0. \\ \tilde{c}_{jk} &= \frac{c_{jt}}{c_{kt}} = \frac{\kappa_j}{\kappa_k} \prod_{\substack{r \in \mathcal{N} \\ r \neq t}} \frac{\rho_j - \rho_r}{\rho_t - \rho_r} \prod_{\substack{r \in \mathcal{N} \\ r \neq t}} \frac{\rho_t - \rho_r}{\rho_k - \rho_r} = \frac{\kappa_j}{\kappa_k} \prod_{\substack{r \in \mathcal{N}' \\ r \neq k}} \frac{\rho_j - \rho_r}{\rho_k - \rho_r}, & j \in \mathcal{P}_0. \\ \tilde{c}_{ts} &= -\frac{c_{ks}}{c_{kt}} = -\frac{\kappa_t}{\kappa_s} \prod_{\substack{r \in \mathcal{N} \\ r \neq s}} \frac{\rho_k - \rho_r}{\rho_s - \rho_r} \prod_{\substack{r \in \mathcal{N} \\ r \neq t}} \frac{\rho_t - \rho_r}{\rho_k - \rho_r} = \frac{\kappa_t}{\kappa_s} \prod_{\substack{r \in \mathcal{N}' \\ r \neq s}} \frac{\rho_t - \rho_r}{\rho_s - \rho_r} & s \in \mathcal{N}_0. \\ \tilde{c}_{tk} &= \frac{1}{c_{kt}} = \frac{\kappa_t}{\kappa_k} \prod_{\substack{r \in \mathcal{N} \\ r \neq t}} \frac{\rho_t - \rho_r}{\rho_k - \rho_r} = \frac{\kappa_t}{\kappa_k} \prod_{\substack{r \in \mathcal{N}' \\ r \neq k}} \frac{\rho_t - \rho_r}{\rho_k - \rho_r}.\end{aligned}$$

Thus the  $\tilde{c}_{js}$  are given by similar expressions in terms of the  $\rho_\ell, \kappa_\ell$  as the  $c_{js}$ , and so the  $\tilde{c}_{js}$  satisfy similar relations to those satisfied by the  $c_{js}$ .

## B Relations for the 6-point function

$$\begin{aligned} V_j A_{ir} - V_i A_{jr} &= -p^2 [k, r], & A_{ir} W_s - A_{is} W_r &= -p^2 \langle i, t \rangle \\ V_j A_{ir} W_s + V_i A_{js} W_r - V_j A_{is} W_r - V_i A_{jr} W_s &= p^2 \bar{A}_{tk} \end{aligned}$$

where

$$\begin{aligned} \bar{A}_{ri} &= \sum_{s \in \mathcal{N}} \langle r, s \rangle [s, i] = - \sum_{j \in \mathcal{P}} \langle r, j \rangle [j, i], & \bar{a}_{ri} &= \frac{1}{p^2} \bar{A}_{ri} \\ \det c &= \frac{\beta}{p^2} \sum_{\substack{i \in \mathcal{P} \\ r \in \mathcal{N}}} V_i W_r (a_{js} a_{kt} - a_{jt} a_{ks}) = \frac{\beta}{(p^2)^2} \sum_{i \in \mathcal{P}} V_i \bar{V}_i \sum_{r \in \mathcal{N}} W_r \bar{W}_r = \beta, \end{aligned}$$

where, as before,  $(i, j, k)$ ,  $(r, s, t)$  are cyclic.

Corresponding to the relations for  $A_{ir}$ , we have

$$\begin{aligned} \bar{A}_{ri} \bar{A}_{sj} - \bar{A}_{si} \bar{A}_{rj} &= p^2 W_t V_k, \\ \bar{A}_{ri} \bar{V}_j - \bar{A}_{rj} \bar{V}_i &= -p^2 \langle r, k \rangle, \\ \bar{W}_s \bar{A}_{ri} - \bar{W}_r \bar{A}_{si} &= -p^2 [t, i], \\ \bar{W}_s \bar{A}_{ri} \bar{V}_j + \bar{W}_r \bar{A}_{sj} \bar{V}_i - \bar{W}_r \bar{A}_{si} \bar{V}_j - \bar{W}_s \bar{A}_{rj} \bar{V}_i &= p^2 A_{kt} \\ \bar{c}_{ri} \bar{c}_{sj} - \bar{c}_{si} \bar{c}_{rj} &= \bar{\beta} c_{kt}. \end{aligned}$$

When  $c_{ir} = 0$ ,

$$\begin{aligned} V_i W_r \beta \bar{c}_{ri} &= p_{jkr}^2, \\ V_i W_r \beta \bar{c}_{sj} &= \langle i, s \rangle [r, j], & V_i W_r \beta \bar{c}_{tk} &= \langle i, t \rangle [r, k], & V_i W_r \beta \bar{c}_{sk} &= \langle i, s \rangle [r, k], & V_i W_r \beta \bar{c}_{tj} &= \langle i, t \rangle [r, j], \\ W_r c_{is} &= \langle i, t \rangle, & W_r c_{it} &= -\langle i, s \rangle, & V_i c_{jr} &= [k, r], & V_i c_{kr} &= -[j, r], \\ V_i W_r \beta \bar{c}_{rj} &= \langle i | P_{rk} | j \rangle, & V_i W_r \beta \bar{c}_{rk} &= \langle i | P_{rj} | k \rangle, & V_i W_r \beta \bar{c}_{si} &= \langle s | P_{ti} | r \rangle, & V_i W_r \beta \bar{c}_{ti} &= \langle t | P_{si} | r \rangle, \\ V_i W_r c_{js} &= [k | P_{si} | t], & V_i W_r c_{kt} &= [j | P_{ti} | s], \end{aligned}$$

using notation defined in (6.12) and (6.13).

When  $\bar{c}_{ir} = 0$ ,

$$\begin{aligned} \bar{V}_i \bar{W}_r \bar{\beta} c_{ir} &= p_{jkr}^2, \\ \bar{V}_i \bar{W}_r \bar{\beta} c_{js} &= [i, s] \langle r, j \rangle, & \bar{V}_i \bar{W}_r \bar{\beta} c_{kt} &= [i, t] \langle r, k \rangle, & \bar{V}_i \bar{W}_r \bar{\beta} c_{ks} &= [i, s] \langle r, k \rangle, & \bar{V}_i \bar{W}_r \bar{\beta} c_{jt} &= [i, t] \langle r, j \rangle, \\ \bar{W}_r \bar{c}_{si} &= -[i, t], & \bar{W}_r \bar{c}_{ti} &= [i, s], & \bar{V}_i \bar{c}_{rj} &= -\langle k, r \rangle, & \bar{V}_i \bar{c}_{rk} &= \langle j, r \rangle, \\ \bar{V}_i \bar{W}_r \bar{\beta} c_{jr} &= [i | P_{rk} | j \rangle, & \bar{V}_i \bar{W}_r \bar{\beta} c_{kr} &= [i | P_{rj} | k \rangle, & \bar{V}_i \bar{W}_r \bar{\beta} c_{is} &= [s | P_{ti} | r \rangle, & \bar{V}_i \bar{W}_r \bar{\beta} c_{it} &= [t | P_{si} | r \rangle, \\ \bar{V}_i \bar{W}_r \bar{c}_{sj} &= -\langle k | P_{si} | t \rangle, & \bar{V}_i \bar{W}_r \bar{c}_{tk} &= -\langle j | P_{ti} | s \rangle, \end{aligned}$$

where  $\bar{\beta} = 1/\beta$ .

## C Relations for the 7-point function

We derive the relations we need to evaluate the 7-point function working directly from the equations

$$\pi_\ell = c_{\ell r} \pi_r + c_{\ell s} \pi_s + c_{\ell t} \pi_t, \quad \ell = i, j, k, l, \quad (\text{C.1})$$

$$-\bar{\pi}_u = \bar{\pi}_i c_{iu} + \bar{\pi}_j c_{ju} + \bar{\pi}_k c_{ku} + \bar{\pi}_l c_{lu}, \quad u = r, s, t. \quad (\text{C.2})$$

If  $c_{kt} = 0$ , from (C.1) with  $\ell = k$ ,

$$c_{kr} = \langle k, s \rangle / \langle r, s \rangle, \quad c_{ks} = \langle k, r \rangle / \langle s, r \rangle, \quad c_{kt} = 0. \quad (\text{C.3})$$

(a) For  $c_{kt} = c_{lr} = 0$ , in addition to (C.3),

$$c_{lr} = 0, \quad c_{ls} = \langle l, t \rangle / \langle s, t \rangle, \quad c_{lt} = \langle l, s \rangle / \langle t, s \rangle. \quad (\text{C.4})$$

and from (C.2) with  $u = r$ ,

$$\bar{\pi}_i c_{ir} + \bar{\pi}_j c_{jr} = -(\bar{\pi}_k \langle k, s \rangle + \bar{\pi}_r \langle r, s \rangle) / \langle r, s \rangle,$$

yielding

$$c_{ir} = [j|P_{kr}|s] / [i, j] \langle r, s \rangle, \quad c_{jr} = [i|P_{kr}|s] / [j, i] \langle r, s \rangle, \quad (\text{C.5})$$

and, similarly,

$$c_{it} = [j|P_{lt}|s] / [i, j] \langle t, s \rangle, \quad c_{jt} = [i|P_{lt}|s] / [j, i] \langle t, s \rangle. \quad (\text{C.6})$$

From (C.1) with  $\ell = i$

$$c_{is} \langle s, t \rangle = \langle i, t \rangle - c_{ir} \langle r, t \rangle, \quad c_{js} \langle s, t \rangle = \langle j, t \rangle - c_{jr} \langle r, t \rangle$$

implying

$$(c_{ir} c_{js} - c_{jr} c_{is}) \langle s, t \rangle = \langle j, t \rangle c_{ir} - \langle i, t \rangle c_{jr} = -\langle t | P_{ij} P_{kr} | s \rangle / [i, j] \langle r, s \rangle$$

so that

$$c_{ir} c_{js} - c_{jr} c_{is} = -\langle t | P_{ij} P_{kr} | s \rangle / [i, j] \langle r, s \rangle \langle s, t \rangle, \quad (\text{C.7})$$

and, similarly,

$$c_{it} c_{js} - c_{jt} c_{is} = -\langle r | P_{ij} P_{lt} | s \rangle / [i, j] \langle s, t \rangle \langle r, s \rangle. \quad (\text{C.8})$$

(b) When  $c_{kt} = c_{it} = 0$ , from (C.1) with  $\ell = k, i$ ,

$$c_{kr} = \langle k, s \rangle / \langle r, s \rangle, \quad c_{ks} = \langle k, r \rangle / \langle s, r \rangle, \quad c_{kt} = 0, \quad (\text{C.9})$$

and

$$c_{ir} = \langle i, s \rangle / \langle r, s \rangle, \quad c_{is} = \langle i, r \rangle / \langle s, r \rangle, \quad c_{it} = 0. \quad (\text{C.10})$$

From (C.2) with  $u = t$ ,

$$c_{jt} = -[t, l] / [j, l], \quad c_{lt} = -[t, j] / [l, j], \quad (\text{C.11})$$

and from (C.1) with  $\ell = j, l$ ,

$$c_{jr} \langle r, s \rangle = \langle s | P_{jt} | l \rangle / [l, j], \quad c_{js} \langle s, r \rangle = \langle r | P_{jt} | l \rangle / [l, j], \quad (\text{C.12})$$

$$c_{lr} \langle r, s \rangle = \langle s | P_{lt} | j \rangle / [j, l], \quad c_{ls} \langle s, r \rangle = \langle r | P_{lt} | j \rangle / [j, l]. \quad (\text{C.13})$$

From (C.1) with  $\ell = i, j$ ,

$$\langle r, s \rangle^2 [l, j] (c_{ir}c_{js} - c_{is}c_{jr}) = \langle i, r \rangle \langle s | P_{jt} | l \rangle - \langle i, s \rangle \langle r | P_{jt} | l \rangle = -\langle r, s \rangle \langle i | P_{jt} | l \rangle$$

so that

$$c_{ir}c_{js} - c_{is}c_{jr} = \langle i | P_{jt} | l \rangle / \langle r, s \rangle [j, l] \quad (\text{C.14})$$

From (C.1) with  $\ell = j, l$ ,

$$(c_{lr}c_{js} - c_{jr}c_{ls}) \langle s, t \rangle = \langle j, t \rangle c_{lr} - \langle l, t \rangle c_{jr} = (\langle s | P_{lt} | j \rangle \langle j, t \rangle + \langle s | P_{jt} | l \rangle \langle l, t \rangle) / [j, l] \langle r, s \rangle,$$

from which it follows that

$$c_{lr}c_{js} - c_{jr}c_{ls} = -p_{jt}^2 / [j, l] \langle r, s \rangle. \quad (\text{C.15})$$

(c) When  $c_{kt} = c_{rs}^{ij} = 0$ , from (C.1) with  $\ell = k$

$$c_{kr} = \langle k, s \rangle / \langle r, s \rangle, \quad c_{ks} = \langle k, r \rangle / \langle s, r \rangle, \quad c_{kt} = 0, \quad (\text{C.16})$$

and with  $\ell = i, j$ ,

$$c_{jr}\pi_i - c_{ir}\pi_j = (c_{jr}c_{it} - c_{ir}c_{jt})\pi_t,$$

so that

$$c_{jr} \langle i, t \rangle = c_{ir} \langle j, t \rangle, \quad c_{js} \langle i, t \rangle = c_{is} \langle j, t \rangle. \quad (\text{C.17})$$

From (C.2) with  $u = r$ ,

$$-[l, r] = [l, i]c_{ir} + [l, j]c_{jr} + [l, k] \langle k, s \rangle / \langle r, s \rangle,$$

implying

$$-\langle s | P_{kr} | l \rangle / \langle r, s \rangle = [l, i]c_{ir} + [l, j]c_{jr} = \langle t | P_{ij} | l \rangle c_{ir} / \langle i, t \rangle,$$

so that

$$c_{ir} = -\frac{\langle s | P_{kr} | l \rangle \langle i, t \rangle}{\langle t | P_{ij} | l \rangle \langle r, s \rangle}, \quad c_{jr} = -\frac{\langle s | P_{kr} | l \rangle \langle j, t \rangle}{\langle t | P_{ij} | l \rangle \langle r, s \rangle}, \quad c_{jr}c_{it} - c_{ir}c_{jt} = \frac{\langle s | P_{kr} | l \rangle \langle i, j \rangle}{\langle t | P_{ij} | l \rangle \langle r, s \rangle}, \quad (\text{C.18})$$

$$c_{is} = -\frac{\langle r | P_{ks} | l \rangle \langle i, t \rangle}{\langle t | P_{ij} | l \rangle \langle s, r \rangle}, \quad c_{js} = -\frac{\langle r | P_{ks} | l \rangle \langle j, t \rangle}{\langle t | P_{ij} | l \rangle \langle s, r \rangle}, \quad c_{js}c_{it} - c_{is}c_{jt} = \frac{\langle r | P_{ks} | l \rangle \langle i, j \rangle}{\langle t | P_{ij} | l \rangle \langle s, r \rangle}, \quad (\text{C.19})$$

and, from (C.1) with  $\ell = i$ ,

$$c_{it} \langle t | P_{ij} | l \rangle \langle r, s \rangle \langle i, t \rangle = -\langle s | P_{kr} | l \rangle \langle i, t \rangle \langle r, i \rangle + \langle r | P_{ks} | l \rangle \langle i, t \rangle \langle s, i \rangle = -\langle r, s \rangle \langle i | P_{rks} | l \rangle \langle i, t \rangle$$

so that

$$c_{it} = \frac{\langle i | P_{jt} | l \rangle}{\langle t | P_{ij} | l \rangle}, \quad c_{jt} = \frac{\langle j | P_{it} | l \rangle}{\langle t | P_{ij} | l \rangle}. \quad (\text{C.20})$$

From (C.2) with  $u = t$ ,

$$\langle t | P_{ij} | l \rangle [t, l] c_{lt} = -[t, i]c_{it} - [t, j]c_{jt} = -[t, i] \langle i | P_{jt} | l \rangle - [t, j] \langle j | P_{it} | l \rangle$$

implying

$$c_{lt} = \frac{p_{ijt}^2}{\langle t|P_{ij}|l\rangle}. \quad (\text{C.21})$$

From (C.1) with  $\ell = l$ ,

$$\begin{aligned} \langle t|P_{ij}|l\rangle c_{lr} \langle r, s\rangle &= \langle l, s\rangle \langle t|P_{ij}|l\rangle - c_{lt} \langle t, s\rangle \langle t|P_{ij}|l\rangle \\ &= \langle l, s\rangle \langle t|P_{ij}|l\rangle - p_{ijt}^2 \langle t, s\rangle = -\langle t|P_{ij}P_{kr}|s\rangle \end{aligned}$$

so that

$$c_{lr} = -\frac{\langle t|P_{ij}P_{kr}|s\rangle}{\langle r, s\rangle \langle t|P_{ij}|l\rangle}, \quad c_{ls} = -\frac{\langle t|P_{ij}P_{ks}|r\rangle}{\langle s, r\rangle \langle t|P_{ij}|l\rangle}. \quad (\text{C.22})$$

From (C.1) with  $\ell = j, l$ ,

$$(c_{jr}c_{ls} - c_{js}c_{lr}) \langle r, t\rangle = c_{ls} \langle j, t\rangle - c_{js} \langle l, t\rangle$$

implying

$$c_{jr}c_{ls} - c_{js}c_{lr} = -\frac{p_{krs}^2 \langle j, t\rangle}{\langle r, s\rangle \langle t|P_{ij}|l\rangle}. \quad (\text{C.23})$$

(d) When  $c_{lr} = c_{jr} = 0$ , proceeding as in (b), we have

$$c_{lr} = 0, \quad c_{ls} = \langle l, t\rangle / \langle s, t\rangle, \quad c_{lt} = \langle l, s\rangle / \langle t, s\rangle, \quad (\text{C.24})$$

$$c_{jr} = 0, \quad c_{js} = \langle j, t\rangle / \langle s, t\rangle, \quad c_{jt} = \langle j, s\rangle / \langle t, s\rangle, \quad (\text{C.25})$$

$$c_{ir} = -[r, k] / [i, k], \quad c_{kr} = -[r, i] / [k, i], \quad (\text{C.26})$$

$$c_{is} \langle s, t\rangle = \langle t|P_{ir}|k\rangle / [k, i], \quad c_{it} \langle t, s\rangle = \langle s|P_{ir}|k\rangle / [k, i], \quad (\text{C.27})$$

$$c_{ks} \langle s, t\rangle = \langle t|P_{kr}|i\rangle / [i, k], \quad c_{kt} \langle t, s\rangle = \langle s|P_{kr}|i\rangle / [i, k], \quad (\text{C.28})$$

$$c_{it}c_{js} - c_{is}c_{jt} = \langle j|P_{ir}|k\rangle / \langle s, t\rangle [i, k], \quad (\text{C.29})$$

$$c_{is}c_{kt} - c_{it}c_{ks} = p_{ikr}^2 / [i, k] \langle s, t\rangle. \quad (\text{C.30})$$

(e) When  $c_{lr} = c_{st}^{ij} = 0$ , proceeding as in (c), we have

$$c_{lr} = 0, \quad c_{ls} = \langle l, t\rangle / \langle s, t\rangle, \quad c_{lt} = \langle l, s\rangle / \langle t, s\rangle, \quad (\text{C.31})$$

$$c_{it} = -\frac{\langle s|P_{lt}|k\rangle \langle i, r\rangle}{\langle r|P_{ij}|k\rangle \langle t, s\rangle}, \quad c_{jt} = -\frac{\langle s|P_{lt}|k\rangle \langle j, r\rangle}{\langle r|P_{ij}|k\rangle \langle t, s\rangle}, \quad c_{jt}c_{ir} - c_{it}c_{jr} = \frac{\langle s|P_{lt}|k\rangle \langle i, j\rangle}{\langle r|P_{ij}|k\rangle \langle t, s\rangle}, \quad (\text{C.32})$$

$$c_{is} = -\frac{\langle t|P_{ls}|k\rangle \langle i, r\rangle}{\langle r|P_{ij}|k\rangle \langle s, t\rangle}, \quad c_{js} = -\frac{\langle t|P_{ls}|k\rangle \langle j, r\rangle}{\langle r|P_{ij}|k\rangle \langle s, t\rangle}, \quad c_{js}c_{ir} - c_{is}c_{jr} = \frac{\langle t|P_{ls}|k\rangle \langle i, j\rangle}{\langle r|P_{ij}|k\rangle \langle s, t\rangle}, \quad (\text{C.33})$$

$$c_{ir} = \frac{\langle i|P_{jr}|k\rangle}{\langle r|P_{ij}|k\rangle}, \quad c_{jr} = \frac{\langle j|P_{ir}|k\rangle}{\langle r|P_{ij}|k\rangle}, \quad (\text{C.34})$$

$$c_{kr} = \frac{p_{ijr}^2}{\langle r|P_{ij}|k\rangle}, \quad (\text{C.35})$$

$$c_{kt} = -\frac{\langle r|P_{ij}P_{lt}|s\rangle}{\langle t, s\rangle \langle r|P_{ij}|k\rangle}, \quad c_{ks} = -\frac{\langle r|P_{ij}P_{ls}|t\rangle}{\langle s, t\rangle \langle r|P_{ij}|k\rangle}, \quad (\text{C.36})$$

$$c_{jt}c_{ks} - c_{js}c_{kt} = -\frac{p_{lst}^2 \langle j, r\rangle}{\langle t, s\rangle \langle r|P_{ij}|k\rangle}. \quad (\text{C.37})$$

(f) When  $c_{is} = c_{js} = 0$ , also proceeding as in (b), we have

$$c_{ir} = \langle i, t \rangle / \langle r, t \rangle, \quad c_{is} = 0, \quad c_{it} = \langle i, r \rangle / \langle t, r \rangle, \quad (\text{C.38})$$

$$c_{jr} = \langle j, t \rangle / \langle r, t \rangle, \quad c_{js} = 0, \quad c_{jt} = \langle j, r \rangle / \langle t, r \rangle, \quad (\text{C.39})$$

$$c_{ks} = -[s, l] / [k, l], \quad c_{ls} = -[s, k] / [l, k], \quad (\text{C.40})$$

$$c_{kr} \langle r, t \rangle = \langle t | P_{ks} | l \rangle / [l, k], \quad c_{kt} \langle t, r \rangle = \langle r | P_{ks} | l \rangle / [l, k], \quad (\text{C.41})$$

$$c_{lr} \langle r, t \rangle = \langle t | P_{ls} | k \rangle / [k, l], \quad c_{lt} \langle t, r \rangle = \langle r | P_{ls} | k \rangle / [k, l], \quad (\text{C.42})$$

$$c_{it}c_{jr} - c_{ir}c_{jt} = -(\langle i, r \rangle \langle j, t \rangle - \langle j, r \rangle \langle i, t \rangle) / \langle r, t \rangle^2 = \langle i, j \rangle / \langle t, r \rangle. \quad (\text{C.43})$$

$$(c_{kr}c_{lt} - c_{lr}c_{kt}) \langle r, s \rangle = \langle k, s \rangle c_{lt} - \langle l, s \rangle c_{kt} = (\langle r | P_{ls} | k \rangle \langle k, s \rangle + \langle r | P_{ks} | l \rangle \langle l, s \rangle) / [k, l] \langle t, r \rangle,$$

so that

$$c_{kr}c_{lt} - c_{lr}c_{kt} = -p_{kls}^2 / [k, l] \langle t, r \rangle. \quad (\text{C.44})$$

## References

- [1] N. Arkani-Hamed, F. Cachazo, C. Cheung and J. Kaplan, *The S-matrix in twistor space*, [arXiv:0903.2110](#) [[SPIRES](#)].
- [2] N. Arkani-Hamed, F. Cachazo, C. Cheung and J. Kaplan, *A duality for the S matrix*, [arXiv:0907.5418](#) [[SPIRES](#)].
- [3] N. Berkovits, *An alternative string theory in twistor space for  $N = 4$  super-Yang-Mills*, *Phys. Rev. Lett.* **93** (2004) 011601 [[hep-th/0402045](#)] [[SPIRES](#)].
- [4] N. Berkovits and E. Witten, *Conformal supergravity in twistor-string theory*, *JHEP* **08** (2004) 009 [[hep-th/0406051](#)] [[SPIRES](#)].
- [5] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun. Math. Phys.* **252** (2004) 189 [[hep-th/0312171](#)] [[SPIRES](#)].
- [6] N. Berkovits and L. Motl, *Cubic twistorial string field theory*, *JHEP* **04** (2004) 056 [[hep-th/0403187](#)] [[SPIRES](#)].
- [7] R. Roiban, M. Spradlin and A. Volovich, *Dissolving  $N = 4$  loop amplitudes into QCD tree amplitudes*, *Phys. Rev. Lett.* **94** (2005) 102002 [[hep-th/0412265](#)] [[SPIRES](#)].
- [8] R. Roiban, M. Spradlin and A. Volovich, *A googly amplitude from the B-model in twistor space*, *JHEP* **04** (2004) 012 [[hep-th/0402016](#)] [[SPIRES](#)].
- [9] R. Roiban and A. Volovich, *All googly amplitudes from the B-model in twistor space*, *Phys. Rev. Lett.* **93** (2004) 131602 [[hep-th/0402121](#)] [[SPIRES](#)].
- [10] R. Roiban, M. Spradlin and A. Volovich, *On the tree-level S-matrix of Yang-Mills theory*, *Phys. Rev. D* **70** (2004) 026009 [[hep-th/0403190](#)] [[SPIRES](#)].
- [11] F. Cachazo, P. Svrček and E. Witten, *MHV vertices and tree amplitudes in gauge theory*, *JHEP* **09** (2004) 006 [[hep-th/0403047](#)] [[SPIRES](#)].
- [12] M.L. Mangano and S.J. Parke, *Multi-parton amplitudes in gauge theories*, *Phys. Rept.* **200** (1991) 301 [[hep-th/0509223](#)] [[SPIRES](#)].
- [13] L. Dolan and P. Goddard, *Tree and loop amplitudes in open twistor string theory*, *JHEP* **06** (2007) 005 [[hep-th/0703054](#)] [[SPIRES](#)].



- [14] L. Dolan and P. Goddard, *Current algebra on the torus*, *Commun. Math. Phys.* **285** (2009) 219 [[arXiv:0710.3743](#)] [[SPIRES](#)].
- [15] S.J. Parke and T.R. Taylor, *An amplitude for  $n$  gluon scattering*, *Phys. Rev. Lett.* **56** (1986) 2459 [[SPIRES](#)].
- [16] F.A. Berends, W.T. Giele and H. Kuijf, *On relations between multi - gluon and multigraviton scattering*, *Phys. Lett.* **B 211** (1988) 91 [[SPIRES](#)].
- [17] D.A. Kosower, *Next-to-maximal helicity violating amplitudes in gauge theory*, *Phys. Rev.* **D 71** (2005) 045007 [[hep-th/0406175](#)] [[SPIRES](#)].
- [18] Z. Bern, L.J. Dixon and D.A. Kosower, *All next-to-maximally helicity-violating one-loop gluon amplitudes in  $N = 4$  super-Yang-Mills theory*, *Phys. Rev.* **D 72** (2005) 045014 [[hep-th/0412210](#)] [[SPIRES](#)].
- [19] Z. Bern, V. Del Duca, L.J. Dixon and D.A. Kosower, *All non-maximally-helicity-violating one-loop seven-gluon amplitudes in  $N = 4$  super-Yang-Mills theory*, *Phys. Rev.* **D 71** (2005) 045006 [[hep-th/0410224](#)] [[SPIRES](#)].
- [20] R. Britto, F. Cachazo and B. Feng, *New recursion relations for tree amplitudes of gluons*, *Nucl. Phys.* **B 715** (2005) 499 [[hep-th/0412308](#)] [[SPIRES](#)].
- [21] R. Britto, F. Cachazo, B. Feng and E. Witten, *Direct proof of tree-level recursion relation in Yang- Mills theory*, *Phys. Rev. Lett.* **94** (2005) 181602 [[hep-th/0501052](#)] [[SPIRES](#)].
- [22] R. Penrose, *Twistor algebra*, *J. Math. Phys.* **8** (1967) 345 [[SPIRES](#)].
- [23] V.P. Nair, *A current algebra for some gauge theory amplitudes*, *Phys. Lett.* **B 214** (1988) 215 [[SPIRES](#)].
- [24] A.P. Hodges, *Scattering amplitudes for eight gauge fields*, [hep-th/0603101](#) [[SPIRES](#)].
- [25] A. Hodges, *Eliminating spurious poles from gauge-theoretic amplitudes*, [arXiv:0905.1473](#) [[SPIRES](#)].
- [26] J.M. Drummond and J.M. Henn, *All tree-level amplitudes in  $N = 4$  SYM*, *JHEP* **04** (2009) 018 [[arXiv:0808.2475](#)] [[SPIRES](#)].
- [27] M. Spradlin and A. Volovich, *From twistor string theory to recursion relations*, *Phys. Rev.* **D 80** (2009) 085022 [[arXiv:0909.0229](#)] [[SPIRES](#)].